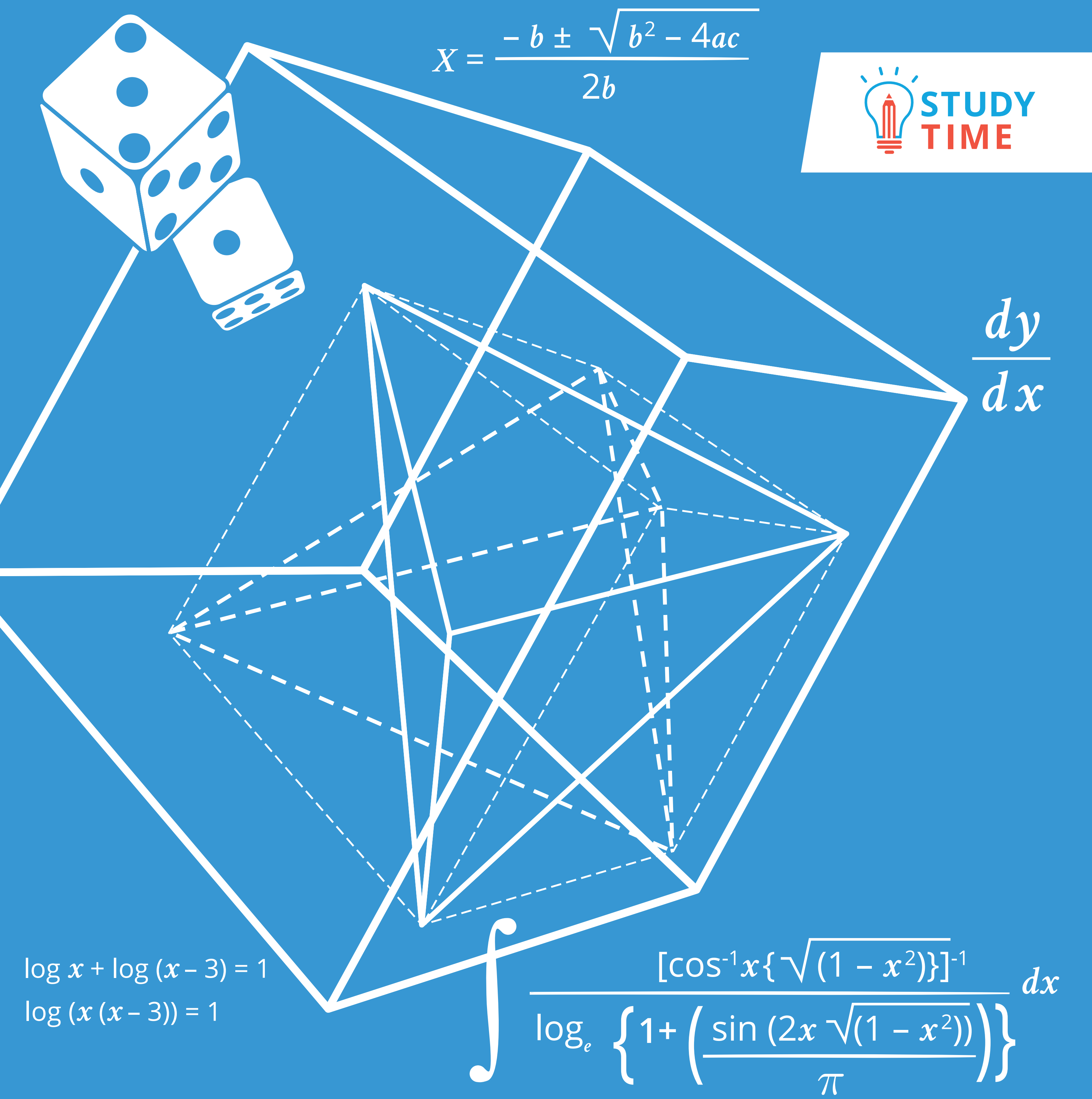
STUDY TIME



LEVEL 2 MATHS

ALGEBRA

NCEA Workbook Answers

Section One
The Foundations

1. Basic Algebra Skills

a. *i.* $2x + 3x = 5x$

ii. $4x + 2y + x^2 + 4y = x^2 + 4x + 6y$ (collecting like terms)

iii. $12x - 3x - 8x = 1x$ (which can simply be written as x)

b. *i.* $a^6 \times a^4$

a^{10}

Rule: $x^n + x^m = x^{n+m}$
If you're confused, try writing out the long form of $a^6 \times a^4$. It breaks down to $(a \times a \times a \times a \times a \times a) \times (a \times a \times a \times a)$. This gives us 'a' multiplied by itself 10 times – which is a^{10}.

ii. $2x^3 \times 3y^2$

$6x^3y^2$

This one's a bit tricky and might have caught you out – we can't actually apply any exponent rules to this one, since the bases of the exponents are different – x and y! However we can simplify 2×3 and pull everything together. Remember that $6x^3y^2$ is actually just the lazy way of writing $6 \times x^3 \times y^2$.

iii. $\frac{x^7}{x^3}$

x^4
Rule: $\frac{x^m}{x^n} = x^{m-n}$
When we divide exponents, we subtract the powers. In this case, we get: $\frac{x^7}{x^3} = x^{7-3} = x^4$.

iv. $\frac{c^{14}}{c^{20}}$

$\frac{1}{c^6}$

If you got c^{-6}, mathematically you're correct $(c^{14-20} = c^{-6})$. But have another look at the question – we asked for positive exponents, which means that we don't want any negative powers. In that case, we apply the exponential rule: $x^{-a} = \frac{1}{x^a}$. Make sure you always double check what the question is asking for to avoid being caught out!

Answers

v. $\frac{14x^{10}}{4x}$

$\frac{7x^9}{2}$ or $\frac{7}{2}x^9$

If we deal with the x values first, we get $\frac{x^{10}}{x^1}$ which is x^{10-1}, which is x^9. But what about the numbers? They can look a bit stressful at first glance, but we can simplify them as if we were asked to simplify a normal fraction. 14 and 4 both have a common factor of 2, so we can cancel the 2 out to get $\frac{7}{2}$. We can't simplify further than that, so we multiply together the number part $(\frac{7}{2})$ and the x part (x^9) to get $\frac{7x^9}{2}$ or $\frac{7}{2}x^9$.

vi. $(a^2)^3$

a^6

Rule: $(x^m)^n = x^{mn}$
When we have something inside brackets, we can expand it out if in doubt. This gives us $(a^2) \times (a^2) \times (a^2)$, since we're multiplying the bracket by itself 3 times. $a^2 \times a^2$ is $a^{2+2} = a^4$, so we have a^4 x a^2. This is a^{4+2} which is a^6.

Sometimes the exponents are super high numbers (or even fractions), so we can't always write out all of the brackets. This is when that exponent rule $(x^m)^n = x^{mn}$ comes into play (and note that you'll get the same answer either way).

vii. $(2x^5)^4$

$16x^{20}$

Rule: $(ax^m)^n = a^n x^{mn}$
So, from our last question, we should remember that to work out the exponent to the power of another exponent, we multiply the powers together. This means for the $(x^5)^4$, we can get x^5 x ^{4}, which is x^{20}.
So how did we get 16? We have to remember to apply the thing outside of the brackets to everything inside the brackets. In this case, we have to apply the power of 4 to 2 as well, which gives us 2^4. 2^4 is a real number which we can work out $(2 \times 2 \times 2 \times 2)$, which gives us 16.
If you're still confused, try expanding the brackets out – this will help you to see why we have to remember to deal with the 2 as well.

viii. $\sqrt[4]{(x^8)}$

x^2

Rule: $\sqrt[n]{x^m} = x^{\frac{m}{n}}$
The first thing we have to deal with is that root sign. If we apply the rule, we can rewrite the original equation as $(x^8)^{\frac{1}{4}}$. Now hopefully this should be looking a bit more familiar – we have an exponent to the power of something else outside the brackets. Just like in the last couple of questions, we can simply multiply the two powers together: $8 \times \frac{1}{4} = \frac{8}{4} = 2$. So our resulting power is 2, so $(x^8)^{\frac{1}{4}} = x^2$.

ix. $\frac{3x^2y^2z^4}{6x^3y^4z}$

$\frac{z^3}{2xy^2}$

(remember that when we cancel out, we divide the numbers by the largest common factor. For the variables, we subtract the exponents)

x. $\frac{7a^2b^4c}{12ab^2c^3}$

$\frac{7ab^2}{12c^2}$

c. *i.* $3 + x = 7$

$x = 7 - 3$
$x = 4$

Remember we are always trying to get x on one side and the numbers on the other side.

ii. $3x + 7 = 2x$

$3x - 2x = -7$
$x = -7$

iii. $y + 2 = 7y - 5$

$y - 7y + 2 = -5$
$-6y = -5 - 2$
$-6y = -7$
$y = \frac{-7}{-6} = \frac{7}{6}$

2. Expanding and Factorising

a. *i.* $ab(cd + ef)$

Factorised form.
Brackets are usually a clear indicator that an expression/equation is factorised in some way.

ii. $x^2 + 5x + 4$

Expanded form.
Check: Is there anything in common with the three terms x^2, $5x$ and 4? Are there any brackets? The answer is no so the expression is fully expanded (Note: x^2 and $5x$ both involve variable x but the powers are different so they are not like terms).

Answers

b. *i.* $x^2y^3z - 4x^2y^3z + yz + 3yz + 8$

$-3x^2y^3z + 4yz + 8$

Remember for like terms, you need the same variables with the same powers attached to them. You add and subtract the numbers like you would if the algebraic stuff wasn't there. The 1st and 2nd terms are like terms (both have x^2y^3z) just as the 3rd and 4th terms are like terms (both have yz). There are no other plain numbers so we can't combine 8 with anything.

ii. $a^2b + b^2c^2 - 3ba^2 - 12 + 3c^2b^2 + 7 + 6cb$

$-2a^2b + 4b^2c^2 + 6cb - 5$

This one is a little trickier because the order of the variables is mixed up in some of the terms. Remember that when collecting like terms, the order doesn't matter as long as the variables and powers are the same. So in this case, the like terms are the 1st and 3rd terms (a^2b), 2nd and 5th terms (b^2c^2) and finally the 4th and 6th terms (both plain numbers).

c. *i.* $a^2b^3d^4e + a^4bc^2 - a^3b^2cde$

$a^2b(a^{2-2}b^{3-1}d^4e + a^{4-2}b^{1-1}c^2 - a^{3-2}b^{2-1}cde) = a^2b(b^2d^4e + a^2c^2 - abcde)$

When finding a common factor, we first look for variables that appear in all of the terms – in this case, a and b. Then we take the lowest power of each variable so a^2 and b (note: $b = b^1$). This is our common factor. Then, we divide by the common factor which means we subtract exponents. This sounds tricky but the next couple of lines of working should hopefully make it clearer. Remember that anything to the power of 0 equals 1.

ii. $6y^2z^4 + 3xy^6z^7 - 9x^3yz^6$

$3yz^4(2y + xy^5z^3 - 3x^3z^2)$

In this case, we also have to consider the coefficients. For coefficients, we can just find the greatest number that fits into **all** of the coefficients – in this case, it is 3. We divide each coefficient by 3 to figure out what to put in the brackets. The variables common to each term are y and z and the lowest powers are y and z^4.

iii. $32n^2m - 4n^4m^2 - 2n^8m$

$2n^2m(16 - 2n^2m - n^6)$

d. *i.* $a^2 - 5a + 6$

$(a - 2)(a - 3)$

Remember that all quadratics have the form $ax^2 + bx + c$ and factorise into the form $(x + ?)(x + ?)$, although in this case, we are using a's instead of x's. In this case the coefficient on the first term is 1, so to factorise the quadratic, we find two numbers that add to b, in this case −5, and multiply to c, in this case 6. Sometimes you can guess the combination quite easily but if you get stuck, it is easiest to work out all of the factors of c (ignore the sign for now, we'll get to that in a sec). The factors of 6 are {1, 2, 3, 6}. Now we need to decide if we want to make both numbers positive/negative or if we are going to make one negative and one positive. Since they need to add to a negative number (−5), at least one needs to be negative. But since they multiply to a positive number (6), we must have 2 negatives. Let's start with our first pair of factors, −1 and −6. They add to −7 so they can't be right. Since that doesn't work, we move to our next pair of factors: 2 and 3. −2 and −3 sum to −5, so they're the right numbers! We remember that quadratics factorise as $(a + ?)(a + ?)$ so we sub −2 and −3 in for the ? signs to get $(a - 2)(a - 3)$

ii. $b^2 + 3b + 2$

$(b + 1)(b + 2)$

iii. $c^2 - 7c + 10$

$(c - 5)(c - 2)$

iv. $d^2 - 4d - 12$

$(d - 6)(d + 2)$

v. $e^2 - e - 6$

$(e - 3)(e + 2)$

vi. $f^2 + 7f + 6$

$(f + 1)(f + 6)$

vii. $7g + g^2 + 10$

$(g + 5)(g + 2)$

Don't get fooled if the terms are not in the right order. The standard form for quadratics is $ax^2 + bx + c$ so we can just reorder the terms until the expression looks familiar: $g^2 + 7g + 10$

viii. $18 + h^2 - 9h$

$(h - 6)(h - 3)$

ix. $i^2 - 9$

$(i + 3)(i - 3)$

This is a specific type of quadratic called 'difference of two squares' that can be hard to factorise unless you know the trick, so keep on the lookout for it. This expression can be rewritten as $i^2 + 0i - 9$, so we are looking for a pair of numbers that multiply to −9, and sum to 0. Immediately, you should know that we need a positive and negative of the same number if they sum to 0. And because they multiply to −9, we know that number is 3.

x. $2j^2 - 11j + 14$

$(j - 2)(2j - 7)$

Here the coefficient on the first term does not equal 1 so we need to use a different method than previously. The first step is writing the expression without the middle term (−11j)
$2j^2 + 14$

Next we multiply the coefficients on the 1st and last terms: $2 \times 14 = 28$. We need to find two numbers that add to the middle term (−11) and multiply to 28. This process should be familiar to you after the previous questions. Factors of 28: {1, 2, 4, 7, 14, 28}. −4 and −7 will do the trick as $-4 \times -7 = 28$ and $-4 + -7 = -11$. Now we write our two numbers in the gap we left between the 1st and last terms and we add a j to each one (or whatever variable you are working with)

$2j^2 - 4j - 7j + 14$

We now factorise by grouping. This is exactly the same as factorising using a common factor except that when we group, we have more than one common factor. As you can see, there is no common factor between **all** four terms but there is a common factor between the 1st and 2nd terms, and the 3rd and 4th terms. This gives us:

$2j(j - 2) - 7(j - 2)$

A good way to check your factorising is by looking at the brackets. If they are the same, then you are almost certainly correct. Lastly, we take out the bracket as a common factor. This sounds complicated but really we just write the bracket (j − 2) and bung whatever is left into a second bracket to get our solution: (j − 2)(2j − 7)

xi. $21k^2 - k - 2$

$21k^2 - 7k + 6k - 2$
$7k(3k - 1) + 2(3k - 1)$
$(3k - 1)(7k + 2)$

xii. $3m^2 + 17m - 6$

$3m^2 + 18m - m - 6$
$3m(m + 6) - (m + 6).$
$(m + 6)(3m - 1)$

xiii. $2n^2 + 19n - 10$

$2n^2 + 20n - n - 10$

Answers

$2n(n + 10) - (n + 10)$
$(n + 10)(2n - 1)$

xiv. $16o^2 - 1$

Once again, we can write this as $16o^2 + 0o - 1$. And, since there is no middle term, the 16 in front of the o^2 term is easy to deal with:

$(4o + 1)(4o - 1)$

e. *i.* $x^2 + 4ax + 4a^2$

As suggested, we'll start by ignoring the a:

$x^2 + 4x + 4 = (x + 2)(x + 2)$

Now what do we have to do to make it correct? We need the x terms to have an a coefficient, and the constant term to have an a^2 coefficient. Let's try:

$(x + 2a)(x + 2a)$

If we expand that back out, we'll find it gives the original expression, so it must be right!

ii. $x^2 - 2yx + y^2$

Ignoring the y:

$x^2 - 2x + 1 = (x - 1)(x - 1)$

Now add in the y:

$(x - y)(x - y)$

iii. $a^2 + 3ba - 2a - 6b$

Ignore the b:

$a^2 + 3a - 2a - 6 = (a + 3)(a - 2)$

Now add in the b:

$(a + 3b)(a - 2)$

iv. $-7x^2 + 6xy + y^2$

This one will actually be a lot easier if we look at it the other way around, and ignore the x:

$y^2 + 6y - 7 = (y + 7)(y - 1)$

Now add in the x:

$(y + 7x)(y - x)$

f. *i.* $(x + 1)(x + 2)(x + 3)$

This is basically expanding a quadratic with an extra bracket added in. If you need to review expanding quadratics, take a look at the Studytime Walkthrough Guide. Expand the last two brackets first and then deal with the first bracket. Don't forget to collect like terms.

$(x + 1)(x^2 + 3x + 2x + 6)$
$(x + 1)(x^2 + 5x + 6)$
$x^3 + 5x^2 + 6x + x^2 + 5x + 6$
$x^3 + 6x^2 + 11x + 6$

ii. $(x+2)^3$

Remember that $(x + 2)^3 = (x + 2)(x + 2)(x + 2)$
$(x + 2)(x+ 2)(x + 2)$
$(x + 2)(x^2 + 2x + 2x + 4)$
$(x + 2)(x^2 + 4x + 4)$
$x^3 + 4x^2 + 4x + 2x^2 + 8x + 8$
$x^3 + 6x^2 + 12x + 8$

iii. $(x + 2)(x - 3)^2$

$(x + 2)(x - 3)(x - 3)$
$(x + 2)(x^2 - 3x - 3x + 9)$
$(x + 2)(x^2 - 6x + 9)$
$x^3 - 6x^2 + 9x + 2x^2 - 12x + 18$
$x^3 - 4x^2 -3x + 18$

iv. $(x + 3)(x - 2)(x - 7)$

$(x + 3)(x^2 - 7x - 2x + 14)$
$(x + 3)(x^2 - 9x + 14)$
$x^3 - 9x^2 + 14x + 3x^2 - 27x + 42$
$x^3 - 6x^2 - 13x + 42$

g. *i.* $4(x + 4) = 3$

For all of these questions, think about BEDMAS which tells us we need to start by expanding the brackets. After that, we just require some simple algebra to get our equation into the form $x = ?$.

$4x + 16 = 3$	Expand brackets
$4x = -13$	Subtract 16 from both sides
$x = -\frac{13}{4}$	Divide both sides by 4 to get x by itself

ii. $2(x + 7) - 3(x - 3) = 12$

$2x + 14 - 3x + 9 = 12$	Expand brackets
$-x + 23 = 12$	Collect like terms
$-x = -11$	Subtract 23 from both sides
$x = 11$	Divide by -1 to get x by itself

iii. $x + 4 - (7x - 2) - (x - 3) = 9$

$x + 4 - 7x + 2 - x + 3 = 9$	Expand brackets
$-7x + 9 = 9$	Collect like terms
$-7x = 0$	Subtract 9 from both sides
$x = 0$	Divide by -7 to get x by itself. Remember that 0 divided by anything is still 0.

iv. $4x = 6(x - 2) - (4 - x)$

$4x = 6x - 12 - 4 + x$	Expand brackets
$4x = 7x - 16$	Collect like terms
$-3x = -16$	Subtract $7x$ from both sides
$x = \frac{-16}{-3}$	Divide by $-3x$ to get x by itself
$x = \frac{16}{3}$	Cancel negatives in fraction

h. *i.* $a^2c^2x + 3x = 7$

$x(a^2c^2 + 3) = 7$
$x = \frac{7}{a^2c^2 + 3}$

ii. $xy^2 + xy^3z = 6$

Keep in mind that, since we are looking to isolate x, we don't factorise out all of the common factors (i.e. y^2), we just factorise out x.

$x(y^2 + y^3z) = 6$
$x = \frac{6}{y^2 + y^3z}$

Answers

3. Algebraic Fractions

a. *i.* $\frac{x^2 - 5x}{x - 5}$

$\frac{x(x-5)}{x-5} = x$

ii. $\frac{x^2 + 6x}{x^2 - 36}$

$\frac{x(x+6)}{(x+6)(x-6)} = \frac{x}{x-6}$

iii. $\frac{(x+3)^2}{x^2 - 9}$

$\frac{(x+3)(x+3)}{(x+3)(x-3)} = \frac{x+3}{x-3}$

iv. $\frac{x^2 + 6x + 8}{x + 4}$

$\frac{(x+4)(x+2)}{x+4} = x+2$

v. $\frac{x+7}{x^2 + 10x + 21}$

$\frac{x+7}{(x+3)(x+7)} = \frac{1}{x+3}$

vi. $\frac{x^2 - 36}{x + 7}$

$\frac{(x+6)(x-6)}{x+7}$

There are no common factors between the numerator and denominator so we can't simplify any further.

vii. $\frac{3c - c^2}{c^2}$

$\frac{c(3-c)}{c^2} = \frac{3-c}{c}$

viii. $\frac{2tw^2 - w^3}{6t - 3w}$

$\frac{w^2(2t-w)}{3(2t-w)} = \frac{w^2}{3}$

ix. $\frac{3mn^2}{m^2n - mn^2}$

$\frac{3mnn}{mn(m-n)} = \frac{3n}{m-n}$

x. $\frac{4q^2}{p^2}$

There are no terms in common between the numerator and denominator so this is another fraction that can't be simplified.

b. *i.* $\frac{2}{x} + \frac{3}{2x}$

We need to find a common denominator in order to combine the fractions. To do this, we first find the lowest common multiple for the numbers and include the **highest** power of each variable. Our common denominator here is $2x$.

Now we need to make each fraction have that common denominator. The denominator in $\frac{2}{x}$ is x. We multiply x by 2 to get $2x$. Whatever we do to the bottom we also do to the top, so we also multiply 2 by 2 to get 4. We don't have to do anything to the second term $\frac{3}{2x}$ because it already has the common denominator. Our fraction becomes:

$\frac{4}{2x} + \frac{3}{2x} = \frac{7}{2x}$. (Remember: Once the denominators of the fractions are the same, we can add the numerators by collecting like terms.)

ii. $\frac{1}{xy} + \frac{1}{x}$

Common denominator: xy

$$\frac{1}{xy} + \frac{y}{xy} = \frac{1 + y}{xy}$$

iii. $\frac{1}{xy} + \frac{1}{xz} + \frac{1}{yz}$

Common denominator: xyz

$$\frac{z}{xyz} + \frac{y}{xyz} + \frac{x}{xyz} = \frac{x + y + z}{xyz}$$

iv. $\frac{3x}{4y^2} + \frac{2}{5xy}$

Common denominator: $20xy^2$

$$\frac{15x^2}{20xy^2} + \frac{8y}{20xy^2} = \frac{15x^2 + 8y}{20xy^2}$$

v. $\frac{3}{x + 3} + \frac{2}{x + 2}$

When our denominators have + or − in them, the common denominator will behave a bit differently. We simply multiply the two denominators together, in this case getting $(x + 3)(x + 2)$. Generally if we have brackets in the denominator, we leave them in factorised form.

$$\frac{3(x + 2)}{(x + 3)(x + 2)} + \frac{2(x + 3)}{(x + 3)(x + 2)} = \frac{3x + 6}{(x + 3)(x + 2)} + \frac{2x + 6}{(x + 3)(x + 2)} = \frac{3x + 6 + 2x + 6}{(x + 3)(x + 2)} = \frac{5x + 12}{(x + 3)(x + 2)}$$

vi. $\frac{x + 1}{x + 2} + \frac{x - 3}{3x + 6}$

Common denominator: $(x + 2)(3x + 6)$

$$\frac{(x + 1)(3x + 6)}{(x + 2)(3x + 6)} + \frac{(x - 3)(x + 2)}{(x + 2)(3x + 6)} = \frac{3x^2 + 6x + 3x + 6}{(x + 2)(3x + 6)} + \frac{x^2 + 2x - 3x - 6}{(x + 2)(3x + 6)} = \frac{3x^2 + 9x + 6}{(x + 2)(3x + 6)} + \frac{x^2 - x - 6}{(x + 2)(3x + 6)}$$
$$= \frac{3x^2 + 9x + 6 + x^2 - x - 6}{(x + 2)(3x + 6)} = \frac{4x^2 + 8x}{(x + 2)(3x + 6)} = \frac{4x(x + 2)}{(x + 2)(3x + 6)} = \frac{4x}{3x + 6}$$

Remember to always factorise the numerator if you can, in case the fraction can be simplified further.

vii. $\frac{y}{x - y} + \frac{x}{y - x}$

Common denominator: $(x-y)(y-x)$

$$\frac{y(y - x)}{(x - y)(y - x)} + \frac{x(x - y)}{(x - y)(y - x)} = \frac{y^2 - xy}{(x - y)(y - x)} + \frac{x^2 - xy}{(x - y)(y - x)} = \frac{y^2 - xy + x^2 - xy}{(x - y)(y - x)} = \frac{y^2 + x^2 - 2xy}{(x - y)(y - x)}$$

c. *i.* $\frac{3}{5x} - \frac{1}{10x}$

Subtracting fractions uses the exact same method as we were using before to add fractions. Just be careful with subtraction signs.

Common denominator: $10x$

$$\frac{6}{10x} - \frac{1}{10x} = \frac{6 - 1}{10x} = \frac{5}{10x} = \frac{1}{2x}$$

ii. $\frac{8}{5x} - \frac{4}{15x}$

Common denominator: $15x$

$$\frac{24}{15x} - \frac{4}{15x} = \frac{24-4}{15x} = \frac{20}{15x} = \frac{4}{3x}$$

iii. $\frac{5}{3x^3z^2} - \frac{5x}{7x^2yz^2}$

Common denominator: $21x^3yz^2$

$$\frac{35y}{21x^3yz^2} - \frac{15x^2}{21x^3yz^2} = \frac{35y - 15x^2}{21x^3yz^2} = \frac{5(7y - 3x^2)}{21x^3yz^2}$$

iv. $\frac{1}{x - 1} - \frac{1}{x + 1}$

Common denominator: $(x-1)(x+1)$

$$\frac{1(x + 1)}{(x - 1)(x + 1)} - \frac{1(x - 1)}{(x - 1)(x + 1)} = \frac{x + 1}{(x - 1)(x + 1)} - \frac{(x - 1)}{(x - 1)(x + 1)} = \frac{x + 1-(x - 1)}{(x - 1)(x + 1)} = \frac{(x + 1-x + 1)}{(x - 1)(x + 1)}$$
$$= \frac{2}{(x - 1)(x + 1)}$$

v. $\frac{x+2}{2x-1} - \frac{x-2}{2x+1}$

Common denominator: $(2x - 1)(2x + 1)$

$$\frac{(x+2)(2x+1)}{(2x-1)(2x+1)} - \frac{(x-2)(2x-1)}{(2x-1)(2x+1)} = \frac{2x^2+x+4x+2}{(2x-1)(2x+1)} - \frac{2x^2-x-4x+2}{(2x-1)(2x+1)} = \frac{2x^2+5x+2-2x^2+5x-2}{(2x-1)(2x+1)}$$
$$= \frac{10x}{(2x-1)(2x+1)}$$

vi. $\frac{1}{x^2-4} - \frac{1}{x^2-x-6}$

We start by factorising to get: $\frac{1}{(x+2)(x-2)} - \frac{1}{(x-3)(x+2)}$. $(x+2)$ appears in both denominators but we only need to put it into our common denominator <u>once</u>. We can think about (x+2) as a variable. Remember that when we find the common denominator, we take the highest power of each variable. In this case, the highest power of $(x+2)$ is $(x+2)^1 = (x+2)$. Our common denominator is $(x+2)(x-2)(x-3)$

$$\frac{1(x-3)}{(x+2)(x-2)(x-3)} - \frac{1(x-2)}{(x+2)(x-2)(x-3)} = \frac{x-3}{(x+2)(x-2)(x-3)} - \frac{x-2}{(x+2)(x-2)(x-3)} =$$
$$\frac{x-3-x+2}{(x+2)(x-2)(x-3)} = -\frac{1}{(x+2)(x-2)(x-3)}$$

d. *i.* $\frac{x^2-16}{9x-6} \times \frac{15x-10}{x+4}$

When we multiply fractions, we multiply the numerators together and then we multiply the denominators together. But first, let's try factorising, to make simplifying easier:

$$\frac{x^2-16}{9x-6} \times \frac{15x-10}{x+4} = \frac{(x-4)(x+4)}{3(3x-2)} \times \frac{5(3x-2)}{x+4}$$
$$\frac{(x-4)(x+4)}{3(3x-2)} \times \frac{5(3x-2)}{x+4} = \frac{(x-4)(x+4)5(3x-2)}{3(3x-2)(x+4)}$$

Now, since we factorised, we can easily simplify all of the terms that appear both on the bottom and top of the fraction.

$$\frac{(x-4)(x+4)5(3x-2)}{3(3x-2)(x+4)} = \frac{5(x-4)}{3}$$

ii. $\frac{4x^2-1}{x^3} \times \frac{x^2-7x}{8x+4}$

$$\frac{(2x-1)(2x+1)}{x^3} \times \frac{x(x-7)}{4(2x+1)} = \frac{(2x-1)(2x+1)x(x-7)}{x^3 4(2x+1)} = \frac{(2x-1)(x-7)}{4x^2} = \frac{2x^2-14x-x+7}{4x^2} = \frac{2x^2-15x+7}{4x^2}$$

e. *i.* $\frac{x+5}{x^3} \div \frac{x^2-25}{9x}$

Dividing fractions is exactly the same as multiplying them except that we flip the second fraction. This is called finding the reciprocal. So our problem becomes:

$$\frac{x+5}{x^3} \times \frac{9x}{x^2-25} = \frac{x+5}{x^3} \times \frac{9x}{(x+5)(x-5)} = \frac{(x+5)9x}{x^3(x+5)(x-5)} = \frac{9}{x^2(x-5)}$$

Answers

ii. $\frac{x^2 + 7x + 12}{4x - 2} \div \frac{x + 4}{2}$

$\frac{x^2 + 7x + 12}{4x - 2} \times \frac{2}{x + 4} = \frac{(x + 4)(x + 3)}{2(2x - 1)} \times \frac{2}{(x + 4)} = \frac{(x + 4)(x + 3)2}{2(2x - 1)(x + 4)} = \frac{x + 3}{2x - 1}$

iii. $\frac{9x^2 - 4}{3x - 2} \div \frac{9x + 6}{5x}$

$\frac{9x^2 - 4}{3x - 2} \times \frac{5x}{9x + 6} = \frac{(3x - 2)(3x + 2)}{3x - 2} \times \frac{5x}{3(3x + 2)} = \frac{(3x - 2)(3x + 2)5x}{(3x - 2)3(3x + 2)} = \frac{5x}{3}$

f. i. $\frac{x}{5} = \frac{3}{2}$

Cross multiplying is useful for situations where we have an equation of the form fraction = fraction. We multiply the numerator of the first term by the denominator of the second term, and the denominator of the first term by the numerator of the second term (hence the name "cross multiplying"). We get:

$2x = 15$
$x = \frac{15}{2}$

ii. $\frac{2x}{7} = \frac{3}{14}$

$28x = 21$
$x = \frac{21}{28} = \frac{3}{4}$

iii. $\frac{14}{2x + 1} = \frac{7}{3}$

$42 = 7(2x + 1)$
$42 = 14x + 7$
$35 = 14x$
$x = \frac{35}{14} = \frac{5}{2}$

iv. $\frac{x + 5}{4} = \frac{x}{6}$

$6(x + 5) = 4x$
$6x + 30 = 4x$
$2x + 30 = 0$
$2x = -30$
$x = \frac{-30}{2} = -15$

v. $\frac{7 - 2x}{3} = \frac{x - 9}{2}$

$2(7 - 2x) = 3(x - 9)$
$14 - 4x = 3x - 27$
$14 - 7x = -27$
$-7x = -41$
$x = \frac{-41}{-7} = \frac{41}{7}$

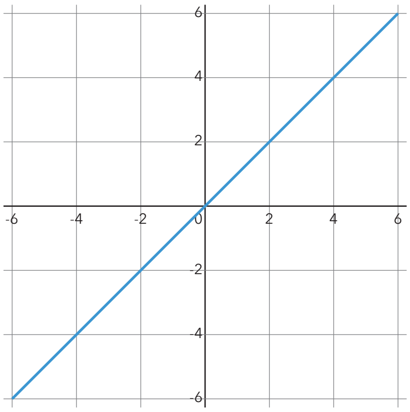
4. Exponents and Logarithms

a. *i.* b

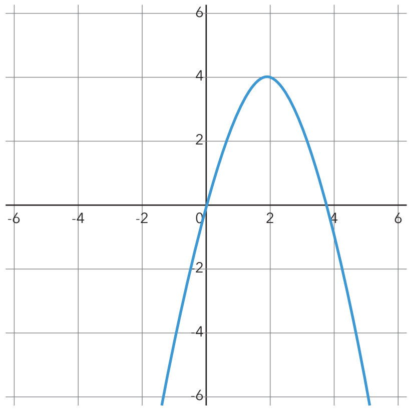
ii. y

iii. x

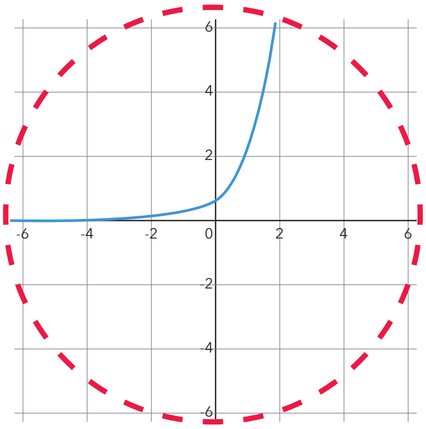
b. i.



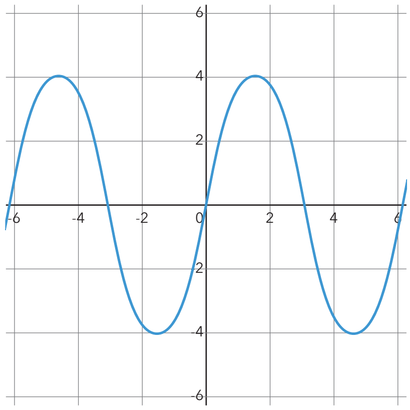
ii.



iii.



iv.



c. *i.* c

ii. d

iii. a

iv. $c^a = d$

d. *i.* $\log_3 2 + \log_3 4$

$\log_3 8$

Rule: $\log_b n + \log_b m = \log_b (n \times m)$

The rule with logs is a counterpart to the exponents rule – so when we add 2 logs together, we multiply the arguments together (as long as the 2 logs have the same base!)

ii. $\log_7 8 - \log_7 2$

$\log_7 4$

Rule: $\log_b n - \log_b m = \log_b(n \div m)$

This is the opposite of the rule we did just before – if we subtract one log from another, you divide the first argument by the second argument (again, making sure the 2 logs have the same base).

iii. $\log_9 1$

Rule: $\log_b 1 = 0$

With exponents, anything to the power of 0 is 1. This rule is the log form equivalent of that (in exponent form it's written as $9^0 = 1$). If your argument is 1, then the power is 0.

iv. $\log_{17} 17$

Rule: $\log_b b = 1$

Again, compare this to exponents. Everything is equal to itself to the power of one. Written in exponent form, it looks like this: $17^1 = 17$.

e. $3\log^2(x)$

Rule: $m\log_b n = \log_b(n^m)$

You can pull anything that's multiplying the log inside the brackets and make it a power (and vice versa).

f. *i* $10^x = 10000$

$x = 4$

The log form is: $\log_{10}(10000) = x$. In your calculator, you should have pressed the log button (which automatically gives you $\log_{10}$), and then typed in 10000 and hit enter.

ii. $e^x = 148$

$x = 5.00$

The log form is $\log_e(148) = x$, which can be written as $\ln(148) = x$.

iii. $e^{2x+1} = 1060$

$x = 2.98$.

This one looks a bit more complicated, but actually we're following the exact same process! The log form is $\log_e(1060) = 2x + 1$.

We put ln(1060) into our calculator, which gives us 6.96602.

$2x + 1 = 6.96602$ (don't round to 2dp yet! You might end up with rounding errors).

Now, just as we would do in algebra normally, we want to get x on a side by itself. So we subtract 1 from both sides to give us $2x = 5.96602$, and then divide both sides by 2 to give us $x = 2.98301$. Now we can round to 2dp, since this is our final answer, which gives us 2.98 (3 rounds down).

g. *i.* $2^x = 42$

$x = 5.39$

First, we take the log of both sides:

$\log 2^x = \log 42$

Note that the base of these logs is 10 so that we can put it into our calculator.

Then, use the log rule $m\log_b n = \log_b(n^m)$ to pull the x out the front.

$x\log 2 = \log 42$

Finally, isolate x on a side by itself. Take the log2 to the other side.

$x = \frac{\log 42}{\log 2}$

Putting this into your calculator should give you the final result of $x = 5.39$.

ii. $5^{x+1} = 70$

$x = 1.64$

Take the log of both sides:

$\log 5^{x+1} = \log 70$

Pull the x+1 out the front. This must be in brackets, as you're treating the whole $(x + 1)$ as one exponent, and so when we follow the rule, we treat it as a single number.

$(x + 1)\log 5 = \log 70$

Isolate the x.

$(x + 1) = \frac{\log 70}{\log 5}$

$x = \frac{\log 70}{\log 5} - 1$

Put it into your calculator, and you should get $x = 1.64$

iii. $90^x = 6^{x^2}$

$x = 2.51$

$\log 90^x = \log 6^{x^2}$

Pull both exponents down to the front of their respective sides.

$x\log 90 = x^2\log 6$

Notice that if we divide both sides by x, we are left with x on only 1 side (which is what we're used to).

$\log 90 = x\log 6$

Now we can isolate x as usual.

$x = \frac{\log 90}{\log 6}$

$x = 2.51$

Answers

5. Quadratics and Polynomials

a. A root is the point at which a function is equal to 0. Graphically, this is where the graph crosses or touches the x–axis. For a quadratic equation in the form $ax^2 + bx + c = 0$, the roots are the same as the solutions to the equation. However, even an expression that isn't set equal to anything, such as $ax^2 + bx + c$, has roots, that will make the expression have a value of 0. For the purposes of NCEA, however, this distinction is minor.

b.
- Distinct, real, rational roots, which is when the graph crosses the x–axis at 2 different points.
- Repeated, real, rational roots, which is when the graph just touches the x–axis at one point .
- Distinct complex conjugate roots, which is when the graph doesn't ever cross the x–axis (it looks like it's floating above it).

In Level 2, you'll only need to deal with the distinct and repeated roots but it is good to keep the third type in mind.

c. Discriminant: $b^2 - 4ac$. It is derived from the quadratic formula. The discriminant is important because it tells us the nature of the roots of an equation.

d. *i.* Discriminant > 0

Distinct, real, rational roots

ii. Discriminant < 0

Complex conjugate roots

iii. Discriminant = 0

Repeated, real, rational roots

e. *i.* $(4y - 6)(y - 2) = 0$

Remember the first step is to set the quadratic equal to zero. The general theory is that one or both of the brackets need to equal zero for the entire equation to equal zero therefore we set both brackets to zero and find two solutions.

$$
\begin{array}{lll}
(4y - 6)(y - 2) = 0 & & \\
4y - 6 = 0 & \text{or} & y - 2 = 0 \\
4y = 6 & \text{or} & y = 2 \\
y = \frac{3}{2} & \text{or} & y = 2
\end{array}
$$

ii. $(x + 2)(x - 2) = 0$

$(x + 2)(x - 2) = 0$

$x + 2 = 0 \quad \text{or} \quad x - 2 = 0$

$x = -2 \quad \text{or} \quad x = 2$

iii. $x^2 = 16$

This is a special case. We can square root both sides immediately but you need to remember to include $\pm$ in order to account for all of the solutions (Remember $\pm$ means $+$ or $-$. For example ± 5 is a shorter way of writing $+5$ or -5)

$x = \sqrt{16}$

$x = \pm 4 = +4 \text{ or } -4$

f. *i.* $x^2 - 6x + 2 = 0$

The quadratic formula is $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. This will be given to you on the formula sheet. To use the formula, we need to identify a, b and c. To do this, we compare our given equation with the standard quadratic $ax^2 + bx + c = 0$. We see that a $= 1$, $b = -6$ and $c = 2$. Plugging in the values gives us:

$x_{1,2} = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(2)}}{2(1)}$

$x_1 = 3 + \frac{\sqrt{28}}{2}$

$x_2 = 3 - \frac{\sqrt{28}}{2}$

ii. $x^2 - 4 = 0$

This one is a little tricky. Because there is no x term, we say that a $= 1$, $b = 0$ and $c = -4$

$x_{1,2} = \frac{-(0) \pm \sqrt{(0)^2 - 4(1)(-4)}}{2(1)}$

$x_{1,2} = \frac{\pm\sqrt{16}}{2}$

$x_1 = \frac{4}{2} = 2$

$x_2 = -\frac{4}{2} = -2$

iii. $2x^2 - 7x + 6 = 0$

$x_{1,2} = \frac{-(-7) \pm \sqrt{-7^2 - 4(2)(6)}}{2(2)}$

$x_{1,2} = \frac{7 \pm \sqrt{1}}{4}$

$x_1 = \frac{7 + 1}{4} = 2$

$x_2 = \frac{7 - 1}{4} = \frac{3}{2}$

Answers

g. For all three cases, the:

Discriminant

$= (4k)^2 - 4(2)(2k^2 + 3k - 11)$

$= 16k^2 - 8(2k^2 + 3k - 11)$

$= 16k^2 - 16k^2 - 24k + 88$

$= -24k + 88$

i. Two real, rational solutions.

We set the discriminant to > 0 to get:

$-24k + 88 > 0$

$-24k > -88$

$k < \frac{-11}{-3}$

$k < \frac{11}{3}$

Remember that the inequality flips when we divide by a negative number.

ii. Repeated roots.

We set the discriminant to $= 0$ to get:

$-24k + 88 = 0$

$-24k = -88$

$k = \frac{-11}{-3}$

$k = \frac{11}{3}$

h. First we put the equation into standard form to get: $6x^2 - mx + 3 = 0$. Then we calculate the discriminant and set it equal to zero.

$(-m)^2 - 4(6)(3) = 0$

$m^2 - 72 = 0$

$m^2 = 72$

$m = \pm\sqrt{72}$

i. *i.* $(x + 2)^2(x - 3)$

Finding cubic roots is the same as for quadratics. We start by setting the expression equal to zero. We need one, two or all of the brackets to equal zero in order to make the entire equation equal zero.

$(x + 2)^2(x - 3) = 0$

$(x + 2)(x + 2)(x - 3) = 0$

$x + 2 = 0 \quad \text{or} \quad x + 2 = 0 \quad \text{or} \quad x - 3 = 0$

$x = -2 \quad \text{or} \quad x = -2 \quad \text{or} \quad x = 3$

ii. $(x-3)(x+4)(x-2)$

$(x-3)(x+4)(x-2)=0$				
$x-3=0$	or	$x+4=0$	or	$x-2=0$
$x=3$	or	$x=-4$	or	$x=2$

iii. $(x+7)^3$

Note that we have the same bracket repeated three times. So instead of expanding the expression to $(x+7)(x+7)(x+7)=0$, we can just solve for $x+7=0$ once. This gives us $x=-7$ which we know will be repeated thrice.

j. *i.* $x^2-6x+7=0$

$x=3\pm\sqrt{2}$

The goal here is to make a new quadratic equation but with numbers that will make our two brackets the same, and then solve for x.

We start by pulling the lonely non–x number to the other side.

$x^2-6x=-7$

We then find the number that is half of 6, and then square it. We add this to the left side since that gives us the quadratic we want, but also add it to the right side so that the two sides of the equation are balanced.

$x^2-6x+9=-7+9$

We then tidy up the numbers on the right side.

$x^2-6x+9=2$

Now we have a quadratic on the left side that we should be able to solve and put into two equal brackets. If this is tricky for you, have a go at revising factorising quadratics.

$(x-3)(x-3)=2$

This can be written as:

$(x-3)^2=2$

How neat and tidy! This is the perfect step to check your working – take a second to expand the equation out, and make sure you get what you began with.

We aren't quite at the solution yet though – we still don't know what x is. Our aim now: to get x by itself on a side. First, take the square root of both sides – and don't forget the plus/minus sign.

$x-3=\pm\sqrt{2}$

Then we have to pull the 3 over to the other side, and that's the answer!

$x=3\pm\sqrt{2}$

ii. $x^2+10x-13=0$

Solution: $x=-5\pm\sqrt{38}$

Working:

$x^2+10x=13$

$x^2+10x+25=13+25$

$x^2+10x+25=38$

$(x+5)(x+5)=38$

$(x+5)^2=38$

$x+5=\pm\sqrt{38}$

$x=-5\pm\sqrt{38}$

iii. $2x^2 - 8x + 18 = 0$

Solution: $x = -2 \pm \sqrt{5}$
This looks a bit more complicated than the other two, but it's okay – just divide the whole thing by 2 and then follow the same process.

$2x^2 - 8x + 18 = 0$

$x^2 - 4x + 9 = 0$

$x^2 - 4x = -9$

$x^2 - 4x + 4 = -9 + 4$

$x^2 - 4x + 4 = 5$

$(x - 2)(x - 2) = 5$

$(x - 2)^2 = 5$

$x - 2 = \pm\sqrt{5}$

$x = 2 \pm \sqrt{5}$

iv. $-3x^2 + 24x + 5 = 0$

Solution: $x = 4 \pm \sqrt{\frac{48}{3}}$
Again, this one looks more complicated – especially because 3 doesn't divide into 5. Follow the same process as last time – divide the whole equation by −3, then follow through.
$-3x^2 + 24x + 5 = 0$
$x^2 - 8x - \frac{5}{3} = 0$
$x^2 - 8x = \frac{5}{3}$
$x^2 - 8x + 16 = \frac{5}{3} + 16$
Simplify the right-hand side by putting everything over a common denominator so that it's easier to work with. $16 \times 3 = 48$ so $16 = \frac{48}{3}$
$x^2 - 8x + 16 = \frac{5}{3} + \frac{48}{3}$
$x^2 - 8x + 16 = \frac{53}{3}$
$(x - 4)(x - 4) = \frac{53}{3}$
$(x - 4)^2 = \frac{53}{3}$
$x - 4 = \pm\sqrt{\frac{53}{3}}$
$x = 4 \pm \sqrt{\frac{53}{3}}$

k. *i.* $x^2 + 7x + 12 > 0$

First, we just find the roots as if it was a usual quadratic equation:
The quadratic factorises to $(x + 3)(x + 4)$, so the roots are:
$x = -3 \text{ or } x = -4$

Then, to get an idea of when the quadratic will be greater than 0, we let x equal values on either side of and between these two roots (any values in the right range will do, not just the ones we chose):

If $x = -5$:
$(-5)^2 + 7(-5) + 12 = 2$
If $x = -3.5$:
$(-3.5)^2 + 7(-3.5) + 12 = -0.25$
If $x = -2$:
$(-2)^2 + 7(-2) + 12 = 2$
So, the quadratic is > 0 outside of these two roots. Our solution is therefore:
$x < -4, x > -3$

ii. $x^2 - 9x + 8 \leq 0$

The quadratic factorises to $(x - 8)(x - 1)$, so the roots are:
$x = 1 \quad \text{or} \quad x = 8$
If $x = 0$:
$(0)^2 - 9(0) + 8 = 8$
If $x = 4$:
$(4)^2 - 9(4) + 8 = -12$

If $x = 9$:
$(9)^2 - 9(9) + 9 = 9$
So the quadratic is ≤ 0 within these two roots. Our solution is therefore:
$1 \leq x \leq 8$

iii. $-2x^2 - 6x - 4 < 0$

This quadratic factorises to:
$-2(x^2 + 3x + 2) = -2(x + 2)(x + 1)$,
so the roots are:
$x = -2, \qquad \text{or} \quad x = -1$
If $x = -3$:
$-2(-3)^2 - 6(-3) - 4 = -4$
If $x = -1.5$:
$-2(-1.5)^2 - 6(-1.5) - 4 = 0.5$
f $x = 0$:
$-2(0)^2 - 6(0) - 4 = -4$
So the quadratic is < 0 outside of these two roots. Our solution is therefore:
$x < -2, x > -1$

iv. $-5x^2 + 3x + 17 \leq 0$

Since this quadratic will be difficult to factorise, we will solve this using the quadratic formula:

$$x_{1,2} = \frac{-(3) \pm\sqrt{(3)^2 - 4(-5)(17)}}{2(-5)}$$

$$x_{1,2} = \frac{-(3) \pm\sqrt{349}}{-10}$$

$$x_1 = \frac{-3 \pm\sqrt{349}}{-10} = -1.57 \text{ (2 dp)}$$

$$x_2 = \frac{-3 \pm\sqrt{349}}{-10} = 2.17 \text{ (2dp)}$$

If $x = -2$:

$-5(-2)^2 + 3(-2) + 17 = -9$

If $x = 0$:

$-5(0)^2 + 3(0) + 17 = 17$

If $x = 3$:

$-5(3)^2 + 3(3) + 17 = -19$

So the quadratic is ≤ 0 outside of the two roots. Our solution is therefore:

$x \leq -1.57,\ x \geq 2.17$

Section Two
Exam Skills/Mixed Practice

1. Achieved

Question 1:

a. Yes we can rewrite in exponential form. Remember that $\log_a b = c$ is equivalent to $a^c = b$.

$\log_t 243 = 5$ becomes $t^5 = 243$ and $\log_3(4k-1) = 2$ becomes $3^2 = 4k - 1$

b. Both of these problems are exponentials so we just have to apply our fundamental exponential rules. If any of this working is confusing for you, it's a good idea to go back and look at the relevant questions in Section 1.

$t^5 = 243$
$t = \sqrt[5]{243} = 3$

$4k - 1 = 3^2 = 9$
$4k = 10$
$k = \frac{10}{4} = \frac{5}{2}$

Question 2:

a. $2x^2 - 50$
$= 2(x^2 - 25)$ Common factor.
$= 2(x - 5)(x + 5)$ Difference of two squares.

b. $9x^2 - 39x - 30$
$= 3(3x^2 - 13x - 10)$ Common factor.
$= 3(3x^2 + 2x - 15x - 10)$ Factorise manually or use quadratic formula.
$= 3[x(3x + 2) - 5(3x + 2)]$
$= 3(x - 5)(3x + 2)$

c. $\frac{2x^2 - 50}{9x^2 - 39x - 30} = \frac{2(x - 5)(x + 5)}{3(x - 5)(3x + 2)}$

d. $\frac{2(x + 5)}{3(3x + 2)}$

Question 3:

a. $x^2 = 49$

b. $x = \pm\sqrt{49} = \pm 7 = 7, -7$

c. This part is a little tricky but remember that x is the base of the log. We cannot have negative bases so we can eliminate -7 which means $x = 7$.

2. Merit

Question 1:

a. No. It is not in the correct form so we rearrange:

$$(2x - 3)(x + 4) = 13$$
$$2x^2 + 8x - 3x - 12 - 13 = 0$$
$$2x^2 + 5x - 25 = 0$$

b. $ax^2 + bx + c = 0$

c. Comparing answers from (a) and (b), we see that: $a = 2$, $b = 5$ and $c = -25$

d. $x_{1,2} = \dfrac{-(5) \pm \sqrt{(5)^2 - 4(2)(-25)}}{2(2)}$

$x_{1,2} = \dfrac{-5 \pm \sqrt{225}}{4}$

$x_1 = \dfrac{-5 + \sqrt{225}}{4} = \dfrac{5}{2}$

$x_2 = \dfrac{-5 - \sqrt{225}}{4} = -5$

Question 2:

a. The fractions have different denominators. Common denominator: r(r−1)

b. Remember that what we do to the bottom is what we do to the top. We don't have to multiply the denominator of the first fraction by anything to get the common denominator so we don't change the top either. However we have to multiply the denominator of the second fraction by (r−1) to get the common denominator so we do the same to the top. We multiply the third fraction by r.

$$\frac{1}{r(r-1)} - \frac{1(r-1)}{r(r-1)} = \frac{3r}{r(r-1)}$$
$$\frac{1}{r(r-1)} - \frac{r-1}{r(r-1)} = \frac{3r}{r(r-1)}$$

c. Be careful with your signs here

$$\frac{1 - r + 1}{r(r-1)} = \frac{3r}{r(r-1)}$$
$$\frac{-r+2}{r(r-1)} = \frac{3r}{r(r-1)}$$

d. $-r + 2 = 3r$

$-4r = -2$

$r = \dfrac{-2}{-4} = \dfrac{1}{2}$

Question 3:

Answers

a. In this case:

y = the value of the house in 2020 = $400,000

A = the initial value of the house in 2006 (which we don't know yet)

$r = 1 + 0.04 = 1.04$

$t = 2020 - 2006 = 14$

If any of this confuses you, go back to your notes on how basic exponential curves work and how their formula is derived.

b. By substitution:

$A(1.04)^{14} = 400{,}000$

$A = \frac{400{,}000}{(1.04)^{14}} = 230{,}990.0331 \approx \$230{,}990 \text{ (6 sf)}$

Question 4:

a. $(x + 25)(x + 1) = 0$

$x = -25 \quad \text{or} \quad x = -1$

b. $5x^2 + 26x + 5 = 0$

$5x^2 + 25x + 1x + 5 = 0$

$5x(x + 5) + 1(x + 5) = 0$

$(5x + 1)(x + 5) = 0$

$5x + 1 = 0$ or $x + 5 = 0$

$5x = -1$ or $x = -5$

$x = -\frac{1}{5}$ or $x = -5$

c. Solutions of first quadratic are clearly 5 times the solutions of the second quadratic, since $-\frac{1}{5} \times 5 = -1$ and $-5 \times 5 = -25$

3. Excellence

Question 1

a. $(3q + p)(q - 12p) - (2q + p)(q - 16p)$

$= 3q^2 - 36pq + pq - 12p^2 - [2q^2 - 32pq + pq - 16p^2]$

$= 3q^2 - 36pq + pq - 12p^2 - 2q^2 + 32pq - pq + 16p^2$

$= q^2 - 4pq + 4p^2$

Be really careful with your negative signs when you're expanding

b. Yes, it can be factorised: $(q - 2p)(q - 2p)$

We start by ignoring the p:

$q^2 - 4q + 4 = (q - 2)(q - 2)$

And then add the qp back in so the factorising is correct:

$(q - 2p)(q - 2p)$

If you aren't comfortable with this sort of factorisation, see the Expanding and Factorising questions in Section One.

This could also be correctly factorised to $(2p - q)(2p - q)$.

c. Remember that $(a + b)^2 = (a + b)(a + b)$. This looks a lot like our answer from (b) right? That means we can write $(q - 2p)(q - 2p)$ as $(q - 2p)^2$ which is the correct form.
Alternatively, we have $(2p - q)^2$.

d. We compare $(q - 2p)^2$ to $(a + b)^2$. They have identical forms which means we can say $a = q$ and $b = -2p$.

Alternatively, we have $a = -2p$ and $b = q$. Or $a = 2p$ and $b = -q$. Or $a = -q$ and $b = 2p$.

Question 2

a. Quadratic.

b. Quadratic equations have the form: $ax^2 + bx + c = 0$. We want our equation to have the same form.

$$2x^2 + 8x - 3x - 12 = m$$
$$2x^2 + 5x - 12 - m = 0$$

c. The equation has one real solution.

d. Discriminant: $b^2 - 4ac$. For one real solution, discriminant $= 0$.

e. Comparing $2x^2 + 5x - 12 - m = 0$ to the standard quadratic form implies that $a = 2$, $b = 5$ and $c = -12 - m$.

Substituting into the discriminant formula, we get:

$$b^2 - 4ac$$
$$= 5^2 - 4(2)(-12 - m)$$
$$= 25 + 96 + 8m$$
$$= 121 + 8m$$

We set this equal to zero.

$$121 + 8m = 0$$
$$8m = -121$$
$$m = -\frac{121}{8}$$

Question 3

a. We are looking at a quadratic equation. We need to expand and simplify the equation to get it into the correct form $ax^2 + bx + c = 0$.

$$x^2 + 5x - 1 - e(x^2 + 1) = 0$$
$$x^2 + 5x - 1 - ex^2 - e = 0$$
$$x^2 - ex^2 + 5x - 1 - e = 0$$
$$(1 - e)x^2 + 5x - (-1 - e) = 0$$

Answers

b. We want values for e when $x^2 + 5x - 1 - e(x^2 + 1) = 0$ has **real solutions**.

We can use the discriminant: $b^2 - 4ac$. For real solutions, $b^2 - 4ac > 0$

Remember that for **two** real solutions, $b^2 - 4ac > 0$ and for one real solution, $b^2 - 4ac = 0$. Since we are just looking for real solutions in general, we have $b^2 - 4ac \geq 0$

c. $5^2 - 4(1 - e)(-1 - e) \geq 0$
$25 - 4(-1 - e + e + e^2) \geq 0$
$25 - 4(-1 + e^2) \geq 0$
$25 + 4 - 4e^2 \geq 0$
$29 - 4e^2 \geq 0$
$29 \geq 4e^2$
$e^2 \leq \frac{29}{4}$

This implies $|e| \leq \sqrt{\frac{29}{4}}$, or $-\sqrt{\frac{29}{4}} \leq e \leq \sqrt{\frac{29}{4}}$

Remember that working with an inequality is exactly the same as working a normal equation except that when you divide by a negative, the inequality sign flips. Also note that when you square root both sides of an equation, you need to have the $\pm$ to ensure you get two answers for your quadratic.

Question 4

a. Let $k = \log_8 x$

b. $6k^2 + 2k - 4 = 0$

c. $6k^2 + 6k - 4k - 4 = 0$
$6k(k + 1) - 4(k + 1) = 0$
$(6k - 4)(k + 1) = 0$
$6k - 4 = 0 \quad \text{or} \quad k + 1 = 0$
$k = \frac{2}{3} \quad \text{or} \quad k = -1$
Or alternatively use the quadratic formula.

d. Since $k = \log_8 x$, we have:

$k = \log_8 x = \frac{2}{3}$. We'll call this equation (1) .
And:
$k = \log_8 x = -1$. We'll call this equation (2).

Equation (1):
$\log_8 x = \frac{2}{3}$
$x = 8^{\frac{2}{3}} = 4$

Equation 2:
$\log_8 x = -1$
$x = 8^{-1} = \frac{1}{8}$

Answers

Question 5

a. Quadratic formula. It will be too difficult to manually factorise these quadratics.

b. $x = \dfrac{-q \pm \sqrt{q^2 - 4pr}}{2}$

c. $x = \dfrac{-q \pm \sqrt{q^{2\wedge} - 4pr}}{2p}$

d. When we compare these two reflections, we see that the solutions of the first equation are $\frac{1}{p}$ times the solutions of the second equation.

Section Three
Practice Exam

Question One

a. *i.* $2x^2 + 24 = -6 - 19x$

Rearrange the equation:
$2x^2 + 19x + 30 = 0$

Apply the Quadratic Formula (or solve manually):
$x_{1,2} = \frac{-19 \pm \sqrt{19^2 - 4(2)(30)}}{2(2)} = \frac{-19 \pm \sqrt{121}}{4} = \frac{-19 \pm 11}{4}$
$x_1 = -2$
$x_2 = -\frac{15}{2}$

ii. $\frac{5}{x^2} + \frac{2}{x} = 3$

Multiply everything by x^2 to get rid of the fractions:
$5 + 2x = 3x^2$

Rearrange the equation:
$3x^2 - 2x - 5 = 0$

Apply the Quadratic Formula (or solve manually):
$x_{1,2} = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(-5)}}{2(3)}$
$x_{1,2} = \frac{2 \pm \sqrt{64}}{6}$
$x_1 = \frac{2+8}{6} = \frac{5}{3}$
$x_2 = \frac{2-8}{6} = -1$

b. For the drug to completely leave John's system, P = 0

Substitute the known values into the equation:
$0.25t^2 - 5t + 1.5 = 0$

Solve the equation using the quadratic formula:
$x_{1,2} = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(0.25)(1.5)}}{2(0.25)}$
$x_{1,2} = \frac{5 \pm \sqrt{23.5}}{0.5}$
$x_1 = 19.70 \text{ (2dp)}$
$x_2 = 0.30 \text{ (2dp)}$

Justification:
We choose $x = 19.70$ because at $x = 0.30$, the drug has not even entered the bloodstream. So it takes 19 hours and $600.7 = 42$ minutes.

c. Substitute $m = 6$ into the polynomial and simplify:
$(12 - 1)x^2 + (6 + 1)x + (6-4) = 11x^2 + 7x + 2$

If p(x) doesn't touch the x–axis, this implies p(x) has no real roots so apply the discriminant test to the polynomial and confirm the hypothesis.
Discriminant $= b^2 - 4ac$
Discriminant of $p(x) = 7^2 - 4(11)(2) = -39$. -39 is less than 0, so the polynomial has no real roots, hence it does not touch the x–axis.

d. $\dfrac{x^3 - 4x^2 + 4x}{2x^3 - 3x^2 - 2x} = \dfrac{(2x^2 - 5x - 3)(2x^2 - 3x - 4c)}{x^2 - 5x + 6}$

Begin by factorising each numerator and denominator as much as possible:
$x^3 - 4x^2 + 4x = x(x^2 - 4x + 4) = x(x - 2)(x - 2)$
$2x^3 - 3x^2 - 2x = x(2x^2 - 3x - 2) = x(2x + 1)(x - 2)$
$2x^2 - 5x - 3 = (2x + 1)(x-3)$
$2x^2 - 3x - 4c$ has no easy factorisation so is left expanded
$x^2 - 5x + 6 = (x - 3)(x - 2)$

Rewrite the equation in factorised form and simplify it:

$\dfrac{x(x - 2)(x - 2)}{x(2x + 1)(x - 2)} = \dfrac{(2x + 1)(x - 3)(2x^2 - 3x - 4c)}{(x - 3)(x - 2)}$
$\dfrac{x(x - 2)}{x(2x + 1)} = \dfrac{(2x + 1)(2x^2 - 3x - 4c)}{(x - 2)}$
$x = 2x^2 - 3x - 4c$

Solve the quadratic equation:
$2x^2 - 4x - 4c = 0$

$x_{1,2} = \frac{-(-4) \pm \sqrt{4^2 - 4(2)(-4c)}}{2(2)}$

$x_{1,2} = \frac{4 \pm \sqrt{16 + 32c}}{4}$

$x_{1,2} = \frac{4 \pm \sqrt{16(1+2c)}}{4}$

$x_{1,2} = \frac{4 \pm 4\sqrt{1+2c}}{4}$

$x_{1,2} = 1 \pm \sqrt{1 + 2c}$

$x_1 = 1 + \sqrt{1 + 2c}$

$x_2 = 1 - \sqrt{1 + 2c}$

e. For real rational solutions: Discriminant $b^2 - 4ac \geq 0$

$(2k)^2 - 4(3)(2k^2 + 11k - 14) \geq 0$
$4k^2 - 12(2k^2 + 11k - 14) \geq 0$
$4k^2 - 24k^2 - 132k + 168 \geq 0$
$-20k^2 - 132k + 168 \geq 0$

To start, we treat this as a normal equation and solve it using the quadratic formula:

$$k_{1,2} = \frac{132 \pm\sqrt{(-132)^2 - 4(-20)(168)}}{-40}$$

$$k_{1,2} = \frac{132 \pm\sqrt{30864}}{-40}$$

$$k_1 = \frac{132 +\sqrt{30864}}{-40} = -7.69\ (2dp)$$

$$k_2 = \frac{132 -\sqrt{30864}}{-40} = 1.09 (2dp)$$

Now, to solve the inequality, we look at what the function is equal to on either side of and between those roots.

If $k = -8$:

$-20(-8)^2 - 132(-8) + 168 = -56$

If $k = 0$:

$-20(0)^2 - 132(0) + 168 = 168$

If $k = 2$:

$-20(2)^2 - 132(2) + 168 = -176$

So the quadratic is ≥ 0 within these two roots, giving:

$-7.60 \leq k \leq 1.09$

Now, the question asks specifically for positive values, so our final solutions is:

0 < k ≤ 1.09

Question Two

a. We have:

$x = \frac{1}{2}$ and $x = 8$

$x - \frac{1}{2} = 0$ and $x - 8 = 0$

The factorised quadratic is:

$(x - \frac{1}{2})(x - 8)$

$= x^2 - 8x - \frac{1}{2}x + 4$

$= x^2 - \frac{17}{2}x + 4$

Therefore, $p = -\frac{17}{2}$

b. For two equal roots, discriminant $b^2 - 4ac = 0$

$12^2 - 4(2)(4+k) = 0$

$12^2 - 32 - 8k = 0$

$-8k + 112 = 0$

$-8k = -112$

$k = 14$

c. $\frac{4}{x^2 - 16} - \frac{6}{2x - 8}$

$= \frac{4}{(x-4)(x+4)} - \frac{6}{2(x-4)}$

Common denominator is $2(x - 4)(x + 4)$

$= \frac{2 \times 4}{2(x-4)(x+4)} - \frac{6(x+4)}{2(x-4)(x+4)}$

$= \frac{8 - 6x - 24}{2(x-4)(x+4)}$

$= \frac{-6x - 16}{2(x-4)(x+4)}$

$= \frac{-3x - 8}{(x-4)(x+4)}$

d. Area of box:
$(2n - 3)^2 = 4n^2 - 12n + 9$

Area of logo:
$(n + 2)^2 = n^2 + 4n + 4$

Area of border:
$(4n^2 - 12n + 9) - (n^2 + 4n + 4)$
$= 4n^2 - 12n + 9 - n^2 - 4n - 4$
$= 3n^2 - 16n + 5$

Set this equal to 160 and solve for n:
$3n^2 - 16n + 5 = 160$
$3n^2 - 16n - 155 = 0$
$n_{1,2} = \frac{-(-16) \pm \sqrt{(-16)^2 - 4(3)(-155)}}{2(3)}$
$n_{1,2} = \frac{16 \pm \sqrt{2116}}{6}$
$n_1 = \frac{31}{3}$
$n_2 = -5$

The width/height of the logo is $n + 2$. We can see that $n = -5$ will make $n + 2$ negative, which is not possible, so $n = \frac{31}{3}$.

Find the area of the pizza box
$4(\frac{31}{3})^2 - 12(\frac{31}{3}) + 9 = 312 \text{ cm}^2$ (3 significant figures)

Find the cost of making the pizza box in colour:
$312 \times 0.005 = \$1.56$ (2 d.p.)

Donny can afford to make his boxes in colour as the cost will be less than $2.00.

e. Let p be the number of residents who ended up going, therefore $p + 4$ is the initial number of residents going on the trip

Initial cost per resident = $\frac{620}{p+4}$

Final cost per resident = $\frac{620}{p}$

Since the initial price increased by $\$1.20$, we get the equation:
$\frac{620}{p+4} + 1.2 = \frac{620}{p}$

We solve the equation and find p
$\frac{620}{p+4} + 1.2 = \frac{620}{p}$

$620p + 1.2p(p + 4) = 620(p + 4)$

$620p + 1.2p^2 + 4.8p - 620p - 2480 = 0$

$1.2p^2 + 4.8p - 2480 = 0$

$p_{1,2} = \frac{-4.8 \pm \sqrt{4.8^2 - 4(1.2)(-2480)}}{2(1.2)}$

$p_{1,2} = \frac{-4.8 \pm \sqrt{11927.04}}{2.4}$

$p_1 = 43.50 \ (2 \text{ d.p.})$

$p_2 = -47.50 \ (2 \text{ d.p.})$

Clearly we cannot have -47.50 residents so we have p $= 43.50$. Therefore 44 people went on the trip.

We must round up whenever we have people or something else that can't be split up. We can either have 43 people or 44 people but if we choose 43, we have 0.50 of a person we have not accounted for so we must round up to 44.

Question Three

a. *i.* $(216x^4)^{\frac{1}{3}}$

$(216x^4)^{\frac{1}{3}} = 6x^{\frac{4}{3}}$

Remember, when we have an exponent on the outside, we use the rule $(x^m)^n = x^{mn}$, which applies even when the outside exponent is a fraction. So with the x, we multiply together 4 and $\frac{1}{3}$ to get an exponent of $\frac{4}{3}$. Then we have to apply the $\frac{1}{3}$ to 216 too. This gives us $216^{\frac{1}{3}}$, which is 6.

Put it together and we get $6x^{\frac{4}{3}}$

ii. $3\log(y) - \log(8)$

$3\log(y) - \log(8) = \log(\frac{y^3}{8})$

To begin with, we pull the 3 inside the log as an exponent.
$3\log(y) - \log(8) = \log(y^3) - \log(8)$

Then we use the subtraction rule of logs which says
$\log(m) - \log(n) = \log(\frac{m}{n})$. This gives us $\log(\frac{y^3}{8})$.
That's our final answer as we cannot simplify further.

b. $x = \frac{2\log(m)}{2\log(7) - \log(m)}$

To begin with, we apply the log to both sides.
$\log(7^{2x}) = \log(m^{(x+2)})$

We then pull the exponents out to the front of the logs.
$2x\log(7) = (x + 2)\log(m)$

Now it's a good idea to expand our right-hand side so that we can isolate x.
$2x\log(7) = x\log(m) + 2\log(m)$

From here, it's just a case of rearranging to try and get x on a side by itself.
$2x\log(7) - x\log(m) = 2\log(m)$
$x(2\log(7) - \log(m)) = 2\log(m)$ (factorise)
$x = \frac{2\log(m)}{2\log(7) - \log(m)}$ (divide both sides to isolate x)

The x is by itself on the left-hand side, and no x's are on the right-hand side, so x is the subject of the equation like was asked.

c. *i.* n represents the number of years since the population was counted.

ii. The population of little spotted kiwis will have tripled by 2037.

The first thing to notice is that we can use the equation that was given to us above:
$P = P_0 \times 1.07^n$.
Since 2020 is the starting year, P_0 is the population of little spotted kiwi in that 2020 – so $P_0 = 1900$.
In this case P, the population, is 3 times the population in 2020, which was 1900.
$P = 1900 \times 3 = 5700$

We can substitute these into our equation:
$5700 = 1900 \times 1.07^n$

We don't know what n is yet – that's what we're trying to work out! We can divide both sides by 1900, which means that we can follow up by applying the log to both sides (since we are interested in working out an exponent).

$5700 \div 1900 = 1.07^n$
$3 = 1.07^n$
$\log(3) = \log(1.07^n)$

Now we pull n out the front of the log, and rearrange so that n is on a side by itself.
$\log(3) = n\log(1.07)$
$n = \frac{\log(3)}{\log(1.07)}$

Nearly there! We can put this into our calculator since the right-hand side is all real numbers.
$n = 16.24...$

Now, stop. Think. What did we just calculate? n is the number of years since 2020. If we say 16 years, the population won't have tripled by the start of that year, so we instead need 17. 2020 + 17 = 2037, so the population will have tripled by 2037.

iii. If there were 7500 Western brown kiwi in 2015, and 8000 Western brown kiwi in 2020, work out what r is. You can assume the equation will follow the same form of model as in (i).

$r = 1.30$

Firstly, we've been given a pretty big hint – the equation will follow the same form as (i).
$P = 8000$, $P_0 = 7500$, the amount of years between 2020 and 2015 is 5 so $n = 5$.
The one tricky thing is what we put as the rate of change. Taking the above as an example, if the number of kiwi changes by 7% each year, then we multiply the equation by 1.07 for each year of change. So when we want a rate of $r\%$, the expression we use is $1 + \frac{r}{100}$. This puts the percentage into decimal form, and makes sure that we're increasing the original amount by r, not just working out what r is.

Now that we have all that, we can put it into our original equation:
$8000 = 7500 \times (1 + \frac{r}{100})^5$

From here, we rearrange to get r by itself.

$8000 \div 7500 = (1 + \frac{r}{100})^5$	Divide both sides by 7500
$\sqrt[5]{(8000 \div 7500)} = 1 + \frac{r}{100}$	Take the 5th root of both sides
$1.012991368 = 1 + \frac{r}{100}$	
$0.012991368 = \frac{r}{100}$	Subtract 1 from both sides
$r = 1.2991368$	

Now that we have r, we can round to something sensible. We decided to round to 3sf.
r = 1.30
Therefore, the Western kiwi population increases by 1.3% each year.

Answers

