Calculus
ANSWERS

Differentiation and Integration

Gradients
STOP AND CHECK (PAGE 7)

- A gradient tells us how two variables are related. The gradient tells us the slope or rate of change. As an example, we might measure that someone's height increases by exactly 2cm every year. We would say that the rate of change of their height in relation to time (in years) is 2, or that the gradient is 2.
- A linear relationship has a constant gradient, which creates a straight line.
- Because the gradient changes at each point of the parabola. If we pick two coordinates on a parabola and work out the straight-line gradient between them using rise/run, we'd get a different gradient than if we picked a different two points (you could give that a go if you want!) This implies that it's a pretty unreliable method, so instead, we need to discover a new method that will reliably tell us the gradient at a certain point.

Differentiation
STOP AND CHECK (PAGE 9)

- We just want a bunch of powers of *x*. In other words, a polynomial. So, we expand the brackets to find:

$$f(x) = (2x+1)(x+3)$$
$$f(x) = 2x^2 + 6x + 1x + 3$$
$$f(x) = 2x^2 + 7x + 3$$

Then, differentiate using the power rule:

$$f'(x) = 4x + 7$$

- To differentiate this, we differentiate each term individually and then add them together. This gives us:

$$f'(t) = 9t^2 - 4t$$

- This is how we refer to a function. In this case, it has the name f and it does something to x. Remember, it's just a fancy name for an equation. So, $f(x) = x^2$ is how x is related to y when we typically write equations like $y = x^2$.

Gradient at a Certain Point

STOP AND CHECK (PAGE 12)

- Let's rewrite this as a function:

$$f(x) = 2x^2 + 4x + 1$$

Then, find the derivative:

$$f'(x) = 4x + 4$$

We want to find the derivative at the point (2, 1) which means we want to see the derivative give us when $x = 2$:

$$f'(2) = 4(2) + 4$$
$$f'(2) = 12$$

So we say the gradient is 12 at $x = 2$ for our function $f(x)$. For the second part of this question, we substitute $f'(x) = 8$ into our gradient function from the first part of the question, which gives us:

$$f'(x) = 4x + 4$$
$$8 = 4x + 4$$

Now, we rearrange to solve for *x*:

$$4 = 4x$$
$$X = 1$$

Finally, we want to find what out y coordinates is when $x = 1$. To do this, we substitute $x = 1$ into our original equation:

$$y = 2x^2 + 4x + 1$$
$$y = 2(1)^2 + 4(1) + 1$$
$$y = 7$$

So the gradient is 8 at (1, 7)

- When we differentiate a function f(*x*) and we find the derivative, we get another equation which is called the derivative. This equation tells us the gradient of our original function at any value of *x* we like.
- If we have a function f(*x*) and we find the derivative, we label that derivative f′(*x*). The dash is what tells us it's a derivative, and we use the same name (in this case, f, to show what it's the derivative of).

Differentiation (Integration)

STOP AND CHECK (PAGE 14)

- So, we apply the opposite of the power rule, giving:

$$f(x) = \tfrac{1}{2}x^4 - 2x + c$$

To find what c is, we use the fact that we know the equation has to give us 7 when we put in $x = 2$ and then solve for c:

$$7 = \tfrac{1}{2}(2)^4 - 2(2) + c$$
$$7 = 8 - 4 + c$$
$$7 = 4 + c$$

$$c = 3$$

So, the equation of the function is:

$$f(x) = ½x^4 - 2x + 3$$

Differentiation and Integration

QUICK QUESTIONS (PAGE 14)

- We're asked to integrate so we apply the opposite of differentiation, in particular do the opposite of the power rule:

  $$f(x) = 2x^2 + (\tfrac{2}{3})x^3 + c$$

  Since we're not told a point on the line we cannot find what c must be, and so we leave this as our final answer.
- We're asked to find the gradient of this function at a particular point. To do this we want to find the derivative of this function f, giving:

  $$f'(x) = 15x^2 + 4x - 4$$

  We want to know what this equation gives us at $x = 3$:

  $$f'(3) = 15(3)^2 + 4(3) - 4$$
  $$f'(3) = 15(9) + 12 - 4$$
  $$f'(3) = 135 + 8$$
  $$f'(3) = 143$$

  So, the gradient of this function f(x) at $x = 3$ is 143, very steep!

Functions and Graphs

Tangents to Curves

STOP AND CHECK (PAGE 16)

- First find the derivative:

  $$f'(x) = -4x + 6$$

  Next we find what the gradient equation (the derivative) gives us at $x = 2$. This will be the gradient of the tangent line at $x = 2$:

  $$f'(2) = -8 + 6$$
  $$f'(2) = -2$$

  We will also need to find the y value on the equation f(x) at x = 2 to figure out the equation of the line:

  $$f(2) = -2(2)^2 + 6(2) - 1$$
  $$f(2) = -8 + 12 - 1$$
  $$f(2) = 3$$

  Now we use the gradient of the tangent line and the point we know is on the tangent line (2, 3) in the equation for a straight line:

  $$y - 3 = -2(x - 2)$$
  $$y = -2x + 4 + 3$$
  $$y = -2x + 7$$

- A tangent line is a straight line that touches a curve at only one point and has the same gradient as the curve at the point it's touching.

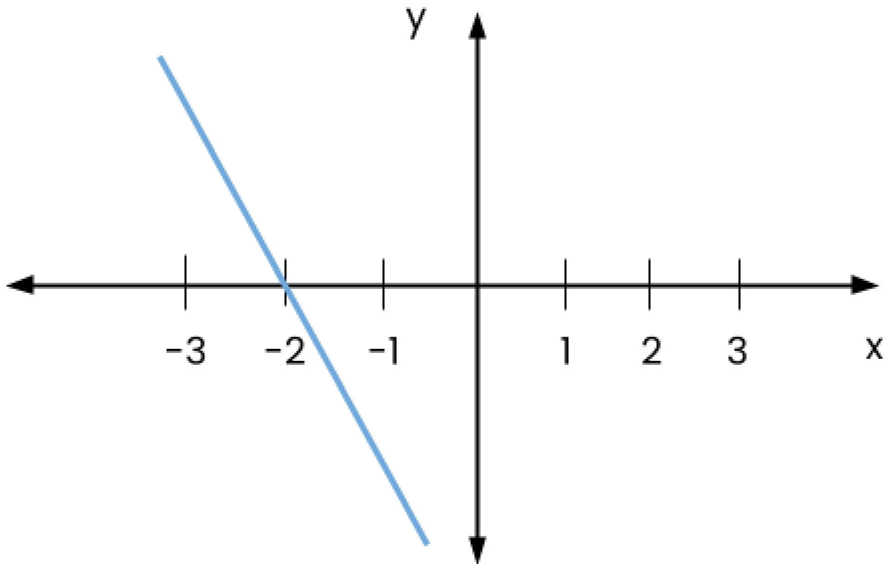
Calculating Turning Points

STOP AND CHECK (PAGE 18)

- A turning point is where the curve swaps between increasing (going up) and decreasing (going down). Since it's going between the two, and it can't swap instantly, the turning point can't be going up or down and so must have a gradient of zero.

Sketching Gradient Functions

STOP AND CHECK (PAGE 22)



- First, we make note of where the curve has a turning point, this tells us where the gradient has to be equal to zero. In this case, that point is at x = -2.
- Next, we think about where the curve is increasing and where it's decreasing. In this case, we notice the curve is increasing to the left of x = -2 which means the gradient must be positive to the left of x = -2. We notice the opposite to the right of x = -2 meaning the gradient must be negative here.
- Finally, we draw the line making sure to have it pass through x = -2 as it has to be equal to zero here and also have the line positive to the left of that point and negative to the right of it.

Function and Graphs

QUICK QUESTIONS (PAGE 23)

- To find the tangent, we need to find the derivative of y:

$$f'(x) = 0.4x$$

Now we substitute in our point of interest to get the gradient:

$$f'(x) = 0.4(7)$$
$$f'(x) = 2.8$$

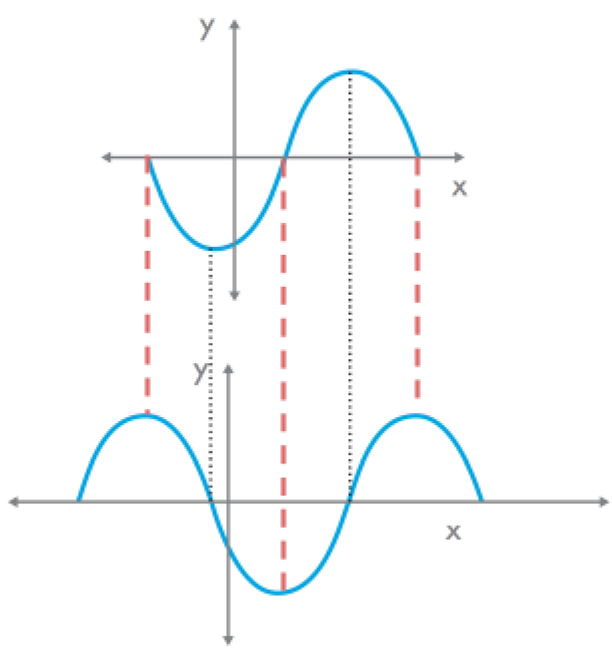
We also need to know the y-value at $x = 7$

$$f(x) = 0.2(7)$$
$$f(x) = 9.8$$

Finally, we use the formula to find the equation of the straight line:

$$y - 9.8 = 2.8(x - 7)$$
$$y = 2.8x - 9.8$$

If you plot the original graph and the tangent you will see it works!



- To sketch the graph of the original function, we do the reverse of sketching the derivative. First, we identify where the derivative is zero. This will tell us where to draw the turning points on the new graph. We identify whether it's a maximum or minimum by finding whether the gradient is going from positive to negative (maximum) or vice versa (minimum.)

Kinematics and Rate of Change

Rate of Change

STOP AND CHECK (PAGE 25)

- We have an equation that tells us the number of words being sung. We want the rate of the words being sung so we find the derivative and then substitute t = 30.

$$N'(t) = 9$$

This equation always gives a value of 9 so we don't even need to substitute.

Kinematics

STOP AND CHECK (PAGE 27)

- To get acceleration from velocity, we must differentiate our velocity equation.

$$\frac{ds}{dt} = a = 2t + 8$$

Now we must substitute our time value in, t = 20.

$$a = -2(20) + 8$$
$$a = -32$$

- Putting it back into context, the bus accelerates at a rate of $-32\ ms^{-2}$ after 20 seconds. Since this is negative, we can instead say that the bus is decelerating at a rate of $32ms^{-2}$.

Integration and Kinematics

STOP AND CHECK (PAGE 28)

- We're interested in the distance the cork travels so we want to integrate the equation for the velocity which will give an equation for distance.

$$d(t) = 2t - 2\ dt$$
$$d(t) = t^2 - 2t + c$$

Unless stated otherwise it makes sense to measure the distance after 0 seconds as 0. So, we can find c by substituting in t = 0, which should give a t-value of 0.

$$d(0) = 0^2 - 2(0) + c$$
$$c = 0$$

So, c = 0. Using the full equation for distance, we can find the distance after 10 seconds:

$$d(10) = 10^2 - 2(10)$$
$$d(10) = 100 - 20$$
$$d(10) = 80$$

So, the cork has travelled 80 metres in 10 seconds.

Kinematics and Related Rates of Change

QUICK QUESTIONS (PAGE 28)

- Finding the rate means we want to find the gradient of the equation at t = 8 so we first find the derivative then substitute t = 8:

$$V'(t) = 12 - \frac{4(t)}{5}$$

$$V'(8) = 12 - \frac{4(8)}{5}$$
$$V'(8) = 5.6$$

So the soil is being removed at a rate of $5.6m^3/h$ after 8 hours.

- We want to know the velocity so we integrate the equation for acceleration:

$$v(t) = 20t - t^2 + c$$

To find c we substitute in t = 0, noting that the velocity when t = 0 is 0.

$$v(0) = 20(0) - 0^2 + c$$
$$c = 0$$

So c = 0. Now, we substitute t = 10 into the full equation for the velocity:

$$v(0) = 20(10) - (10)^2$$
$$v(0) = 200 - 100$$
$$v(0) = 100$$

So the velocity after 10 seconds is $100ms^{-1}$.