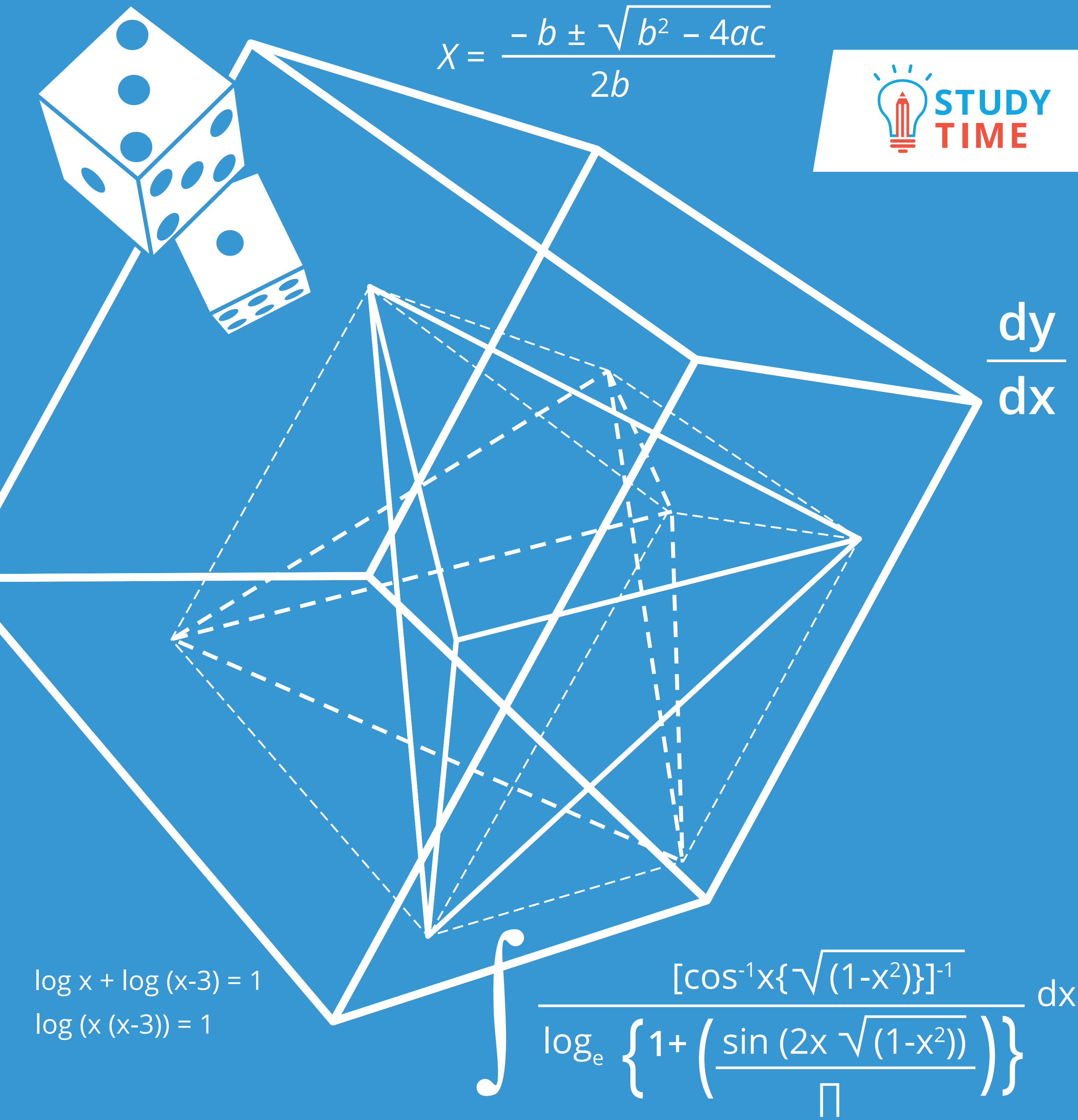


STUDY TIME

LEVEL 2 MATHS

CALCULUS

NCEA Workbook Answers

Section One
Basic Skills

1. Introduction to Functions

a. What is a linear function?

A graph which has a straight line. A linear function has the basic form $y = mx + c$

b. $y = mx + c$ represents a linear equation.

i. What does the y in this equation represent?

The dependent variable of the linear relationship. In other words, it's how far up/down the y–axis the line is at each point, *depending* on what x is.

ii. What does the m in this equation represent, and how could you determine it from a graph?

The gradient of the line. This can be found by calculating the rise/run of the graph

iii. What does the x in this equation represent?

The independent variable, and also how far along the x–axis we are at each point.

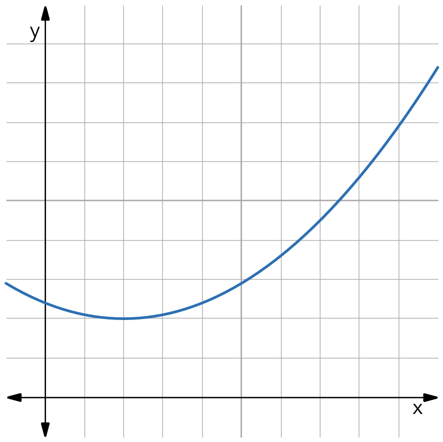
iv. What does the c in this equation represent, and how could you determine it from a graph?

The y–intercept. This is where the line crosses the y–axis (which happens when $x = 0$).

c. Which of the following functions are linear?

The graphs ii), iii), v), and vii) are linear.

i.



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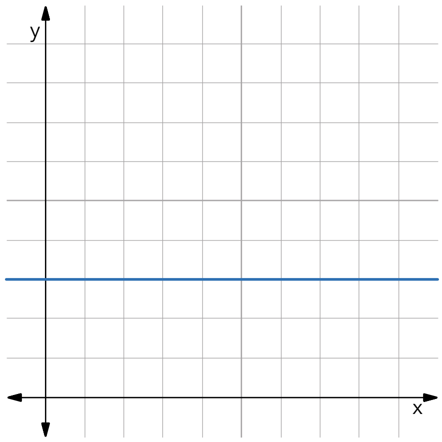

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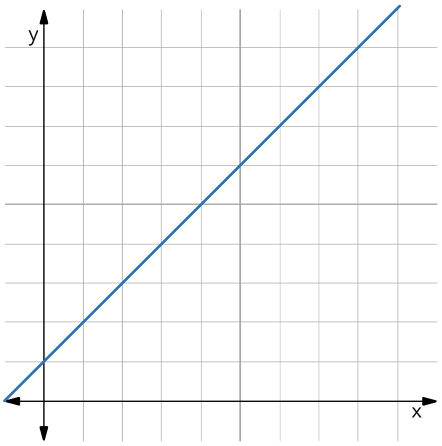

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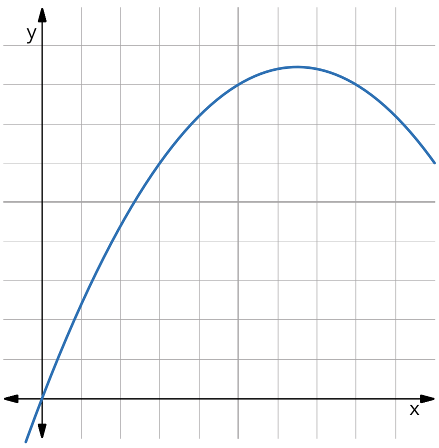

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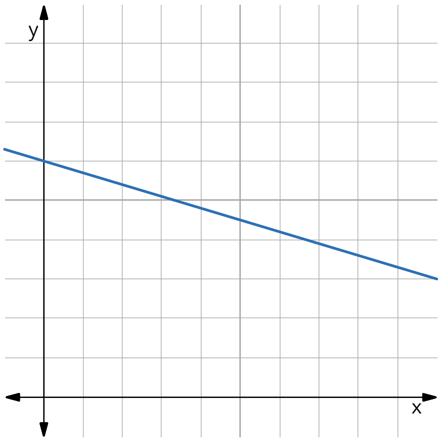

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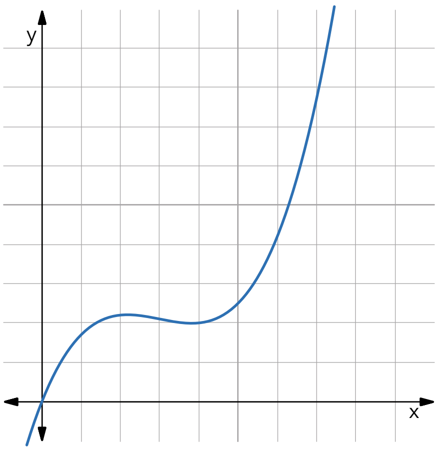

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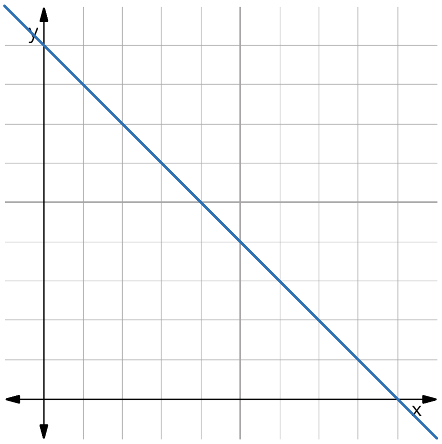

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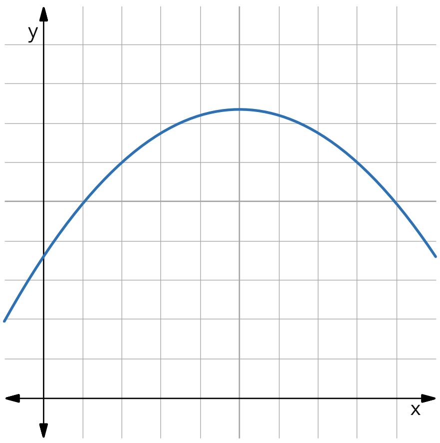
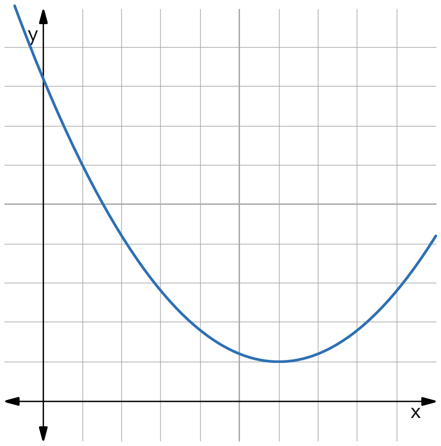


ix.



d. What is a polynomial function?

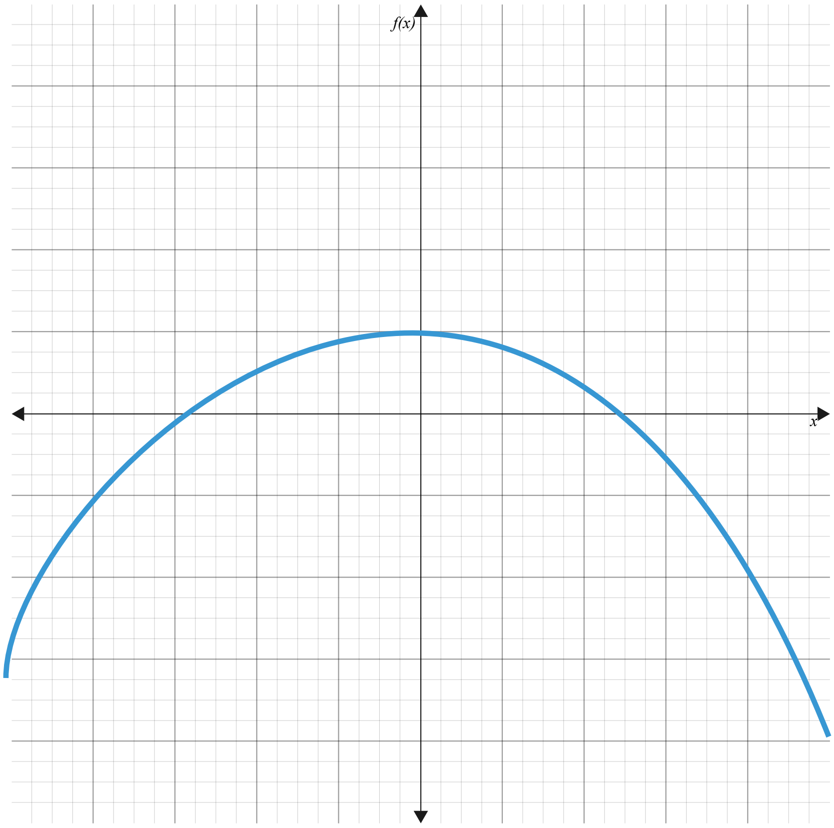
An equation which includes a term of x (the independent variable) raised to a power.

e. What does a quadratic equation look like:

i. In generic equation form?

$y = ax^2 + bx + c$ or $y = (x - a)(x - b)$

ii. On a graph?



is one example of a quadratic. All quadratic graphs have a curved shape, and can be faced up or down.

f. What is a tangent line?

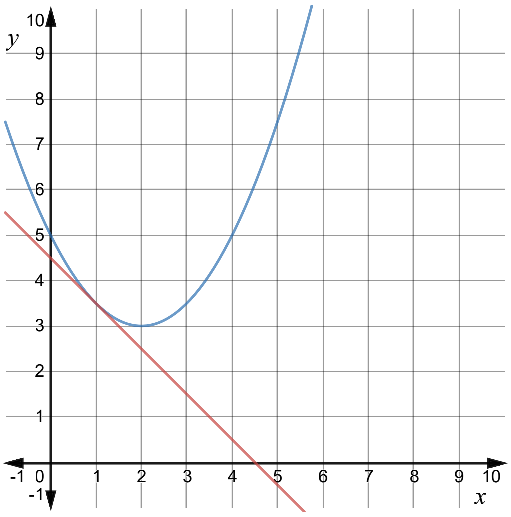
A tangent line is a straight line/ linear graph which intersects with, and has the same gradient as a specific point on a graph.

g. Why are tangent lines more relevant to quadratic equations than linear equations?

Quadratic equations have a changing gradient so each point on the graph has a unique tangent line. As a linear equation has a constant gradient, the tangent line for all points is the same and matches the exact shape of the linear equation.

h. Draw a tangent line on the following graphs at the point $x = 1$

i.



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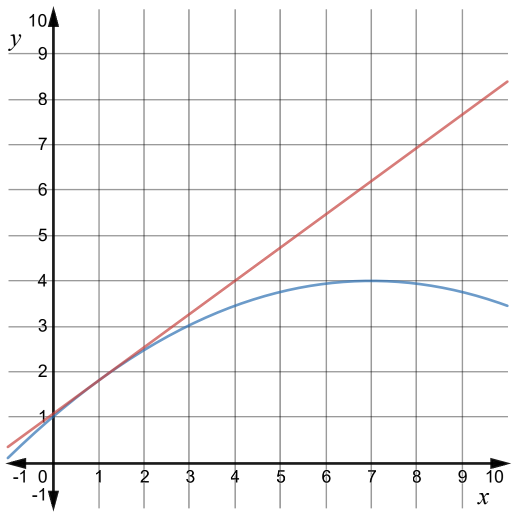

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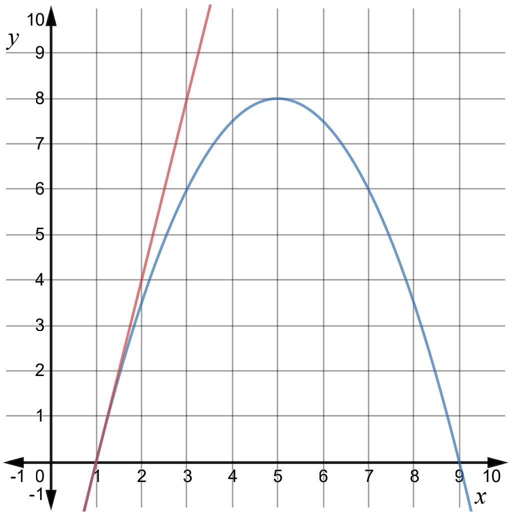

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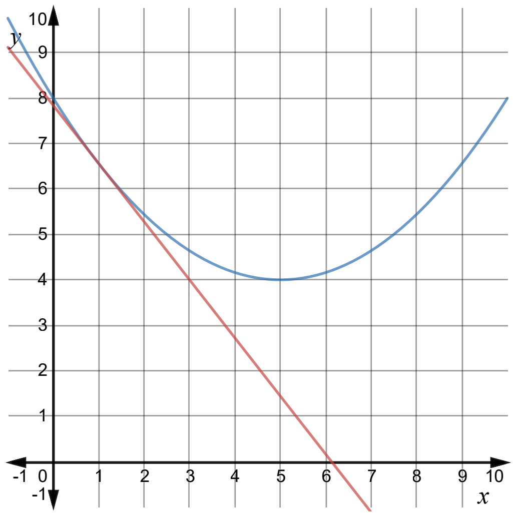


i. What is a normal line?

A normal line is a line which intersects a specific point on a graph and is perpendicular to (at an angle of 90 degrees to) the tangent at that point.
How are the tangent line and normal line of a specific point related?

j. How are the tangent line and normal line of a specific point related?

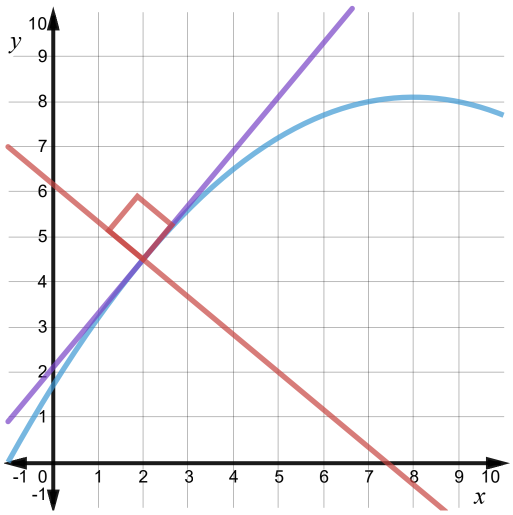
Both intersect the same point on the graph. The normal line is perpendicular to the tangent line at the same point.

k. If you know the gradient of the tangent line, how could you find out the gradient of the normal line?

Perform a 'negative flip'. This means that the normal gradient is given by $-\frac{1}{\text{tangent line gradient}}$. For example, if the tangent line has a slope of 2, then the normal line will have a slope of $-\frac{1}{2}$.

l. Draw a normal line on the following graphs at the point $x = 2$

i.



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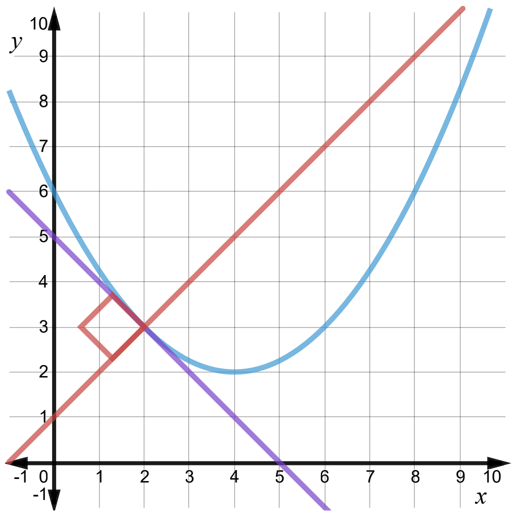

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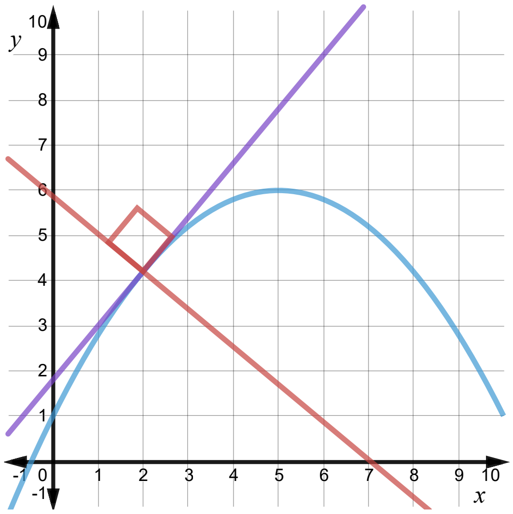

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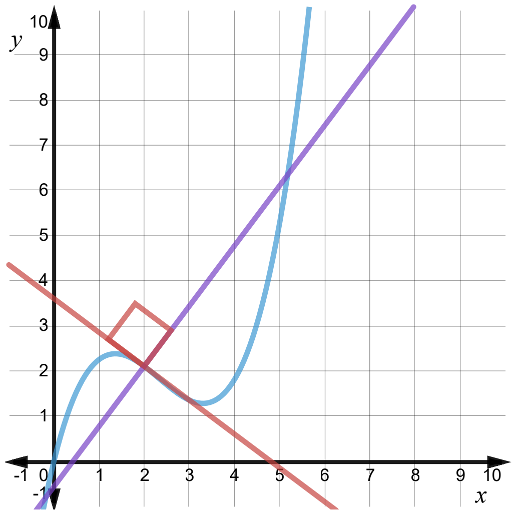


m. What is meant by the expression: 'The function is increasing'?

The gradient has a positive value

n. What is meant by the expression: 'The function is decreasing'?

The gradient has a negative value

o. What are turning points on a graph?

Sections of a graph which have a slope of 0, which are between sections of a graph with an increasing or decreasing slope.

p. Do linear graphs have turning points?

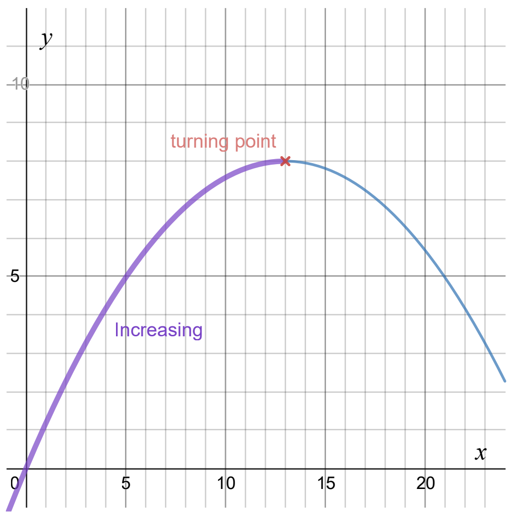
No. Since linear graphs have the same gradient at every point, they can't have a turning point.

q. What is the gradient at a turning point?

Zero

r. On the following graphs indicate the turning points and areas where the function is increasing.

i.



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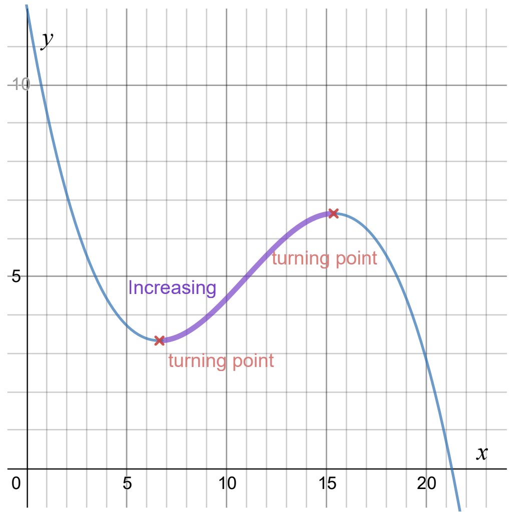

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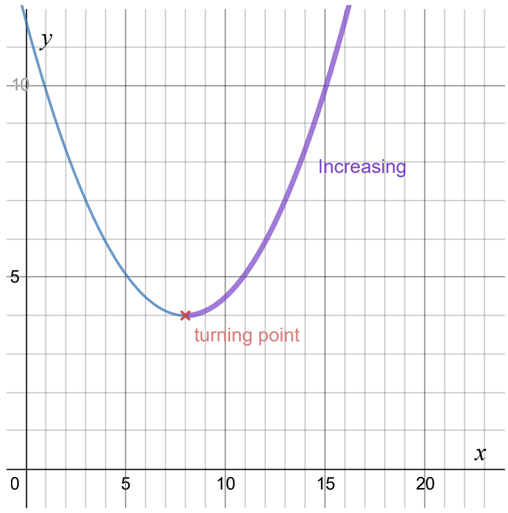

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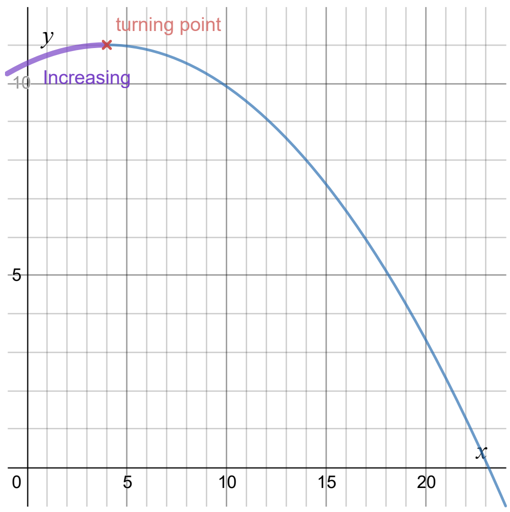

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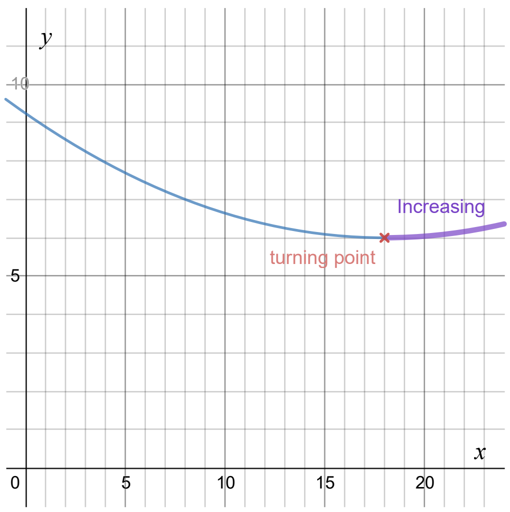


Answers

2. Differentiation Introduction

a. What are the steps required for differentiating?

For each term in the function, you first multiply by the power, and then minus one from the power.

b. What does differentiating a function give you?

The equation of the gradient of the function.

c. What is the gradient of a function? Give an example.

The gradient of a function is the slope of the function if we were to plot it. For a linear function, like $y = 2x + 3$, the slope is constant so the gradient is 2. For a polynomial function where we have x^2 (or an even bigger power), the slope changes at each point of the graph, therefore we need an equation to calculate its value.

d. What are 3 ways we can find the gradient?

1. Linear functions only   Calculate using rise/run
2. Differentiate the equation and substitute in the point you are looking for
3. Draw a tangent line at the point of interest and then use rise/run

e. Functions can be notated as $f(x)$=..... Or y=.....
How do we write the notation of the derivative for these 2 forms of function?

i. $f(x)$ differentiates to:

f'(x)

ii. y differentiates to:

dy/dx

3. Differentiating Simple Functions

a. Finish the following rule: (Hint: you can find it in your formula sheet)

$$\frac{d}{dx}(X^n) = nx^{n-1}$$

b. Put into words what is happening when you use the rule for differentiating above. Give an example of this rule in practice.

When we differentiate a function with just one term, we multiply the function by the power (n) and reduce the power by 1. An example of this is $f(x) = 3x^3$, so the derivative is $f'(x) = 3 \times 3x^{3-1} = 9x^2$.

c. Differentiate the following, making sure to use the correct notation:

i. $y = x^2$

$$\frac{dy}{dx} = 2x$$

ii. $y = x^7$

$$\frac{dy}{dx} = 7x^6$$

Answers

Answers

iii. $y = x$

$y = x^1, \frac{dy}{dx} = 1x^0, \frac{dy}{dx} = 1$

iv. $y = 5x^2$

$\frac{dy}{dx} = 10x$

v. $f(x) = 10x$

$f'(x) = 10$

vi. $f(x) = 3$

$f'(x) = 0$

vii. $f(x) = 12x^2$

$f'(x)=24x$

viii. $f(x) = 13x$

$f'(x) = 13$

ix. $f(x) = 11x^8$

$f'(x)= 88x^7$

x. $y = 2x^6$

$\frac{dy}{dx} = 12x^5$

xi. $y = 3x^4$

$\frac{dy}{dx} = 12x^3$

xii. $y = -2x$

$\frac{dy}{dx} = -2$

xiii. $y = -3x^2$

$\frac{dy}{dx} = -6x$

xiv. $f(x) = -6x^3$

$f'(x)= -18x^2$

xv. $f(x) = -7x^6$

$f'(x) = -42x^5$

xvi. $f(x) = -9$

$f'(x)= 0$

xvii. $y = -15x$

$\frac{dy}{dx} = -15$

xviii. $y = -11x$

$\frac{dy}{dx} = -11$

xix. $f(x) = -3x^8$

$f'(x)= -24x^7$

xx. $y = -4x^6$

$f'(x)= -24x^5$

4. Differentiating Functions with Multiple Terms

a. Differentiate the following:

To differentiate functions with multiple terms, we can simply apply the normal differentiation rule to each individual term

i. $y = 7x^2 + 2x^{11}$

$\frac{dy}{dx} = 14x + 22x^{10}$

ii. $y = 4x^8 + 3 + x$

$\frac{dy}{dx} = 32x^7 + 1$

iii. $y = x + x^2 + x^3 + x^4$

$\frac{dy}{dx} = 1 + 2x + 3x^2 + 4x^3$

iv. $y = 2x^5 - 11x + 13450$

$\frac{dy}{dx} = 10x^4 - 11$

v. $y = 7x + 13x + 12x^2$

$\frac{dy}{dx} = 7 + 13 + 24x,$

$\frac{dy}{dx} = 20 + 24x$

vi. $y = 12 + 5x - 13 + 4x^3 + 7x^2 + 6x$

$\frac{dy}{dx} = 5 + 12x2 + 14x + 6,$

$\frac{dy}{dx} = 12x2 + 14x + 11$

vii. $y = x + x + x$

$y = 3x,$

$\frac{dy}{dx} = 3$

viii. $y = - 3x^4 + 2x^5$

$\frac{dy}{dx} = -12x^3 + 10x^4$

ix. $y = 11x^6 + 5x^4 - 123x + 905$

$\frac{dy}{dx} = 66x^5 + 20x^3 - 123$

x. $f(x) = 19x + 2x^8$

$f'(x) = 19 + 16x^7$

xi. $f(x) = x^2 - 3x^3 + 4x^2 + 12x$

$f'(x) = 2x - 9x^2 + 8x + 12,$

$f'(x) = -9x^2 + 10x + 12$

xii. $f(x) = 18x + 8x^5 + 12$

$f'(x) = 18 + 40x^4$

xiii. $f(x) = 2x^6 + 11x - 7$

$f'(x) = 12x^5 + 11$

xiv. $f(x) = x^8 - x^7$

$f'(x) = 8x^7 - 7x^6$

xv. $f(x) = x^2 - 12x + 5x + 7x + 13$

$f'(x) = 2x - 12 + 5 + 7,$

$f'(x) = 2x + 1$

b. Now let's recap a little bit of algebra.
Expand the following:

i. $y = (x - 1)(x + 3)$

$y = x^2 + 3x - x - 3$

$y = x^2 + 2x - 3$

ii. $y = (x - 3)(x - 7)$

$y = x^2 - 7x - 3x + 21$

$y = x^2 - 10x + 21$

iii. $y = (x + 4)(x + 6)$

$y = x^2 + 6x + 4x + 24$
$y = x^2 + 10x + 24$

iv. $y = (x - 3)^2$

$y = (x-3)(x-3)$
$y = x^2 - 6x + 9$

v. $y = (x + 1)^2$

$y = (x + 1)(x + 1)$
$y = x^2 + 2x + 1$

vi. $y = (2x - 1)(x + 3)$

$y = 2x^2 + 6x - x - 3,$
$y = 2x^2 + 5x - 3$

vii. $y = (11x - 6)(3x - 2)$

$y = 33x^2 + 22x - 18x + 12,$
$y = 33x^2 + 4x + 12$

viii. $y = (11 - x)(3x + 5)$

$y = 33x - 3x^2 - 5x + 55,$
$y = -3x^2 + 28x + 55$

ix. $y = (5 - 2x)(3x + 2)$

$y = 15x - 6x^2 + 10 - 4x,$
$y = -6x^2 + 11x + 10$

c. Differentiate the following. (Hint: use the expansions you did just before)

i. $y = (x - 1)(x + 3)$

$y = x^2 + 2x - 3$
$\frac{dy}{dx} = 2x + 2$

ii. $y = (x - 3)(x - 7)$

$y = x^2 - 10x + 21$
$\frac{dy}{dx} = 2x - 10$

iii. $y = (x + 4)(x + 6)$

$y = x^2 + 10x + 24$
$\frac{dy}{dx} = 2x + 10$

iv. $y = (x - 3)^2$

$y = x^2 - 6x + 9$
$\frac{dy}{dx} = 2x - 6$

v. $y = (x + 1)^2$

$y = x^2 + 2x + 1$
$\frac{dy}{dx} = 2x + 2$

vi. $y = (2x - 1)(x + 3)$

$y = 2x^2 + 5x - 3$
$\frac{dy}{dx} = 4x + 5$

vii. $y = (11x - 6)(3x - 2)$

$y = 66x^2 - 40x + 12$
$\frac{dy}{dx} = 66x - 40$

viii. $y = (11 - x)(3x + 5)$

$y = -3x^2 + 28x + 55$
$\frac{dy}{dx} = -6x + 28$

ix. $y = (5 - 2x)(3x + 2)$

$y = -6x^2 + 11x + 10$
$\frac{dy}{dx} = -12x + 11$

d. Differentiate the following equations:

i. $f(x) = (x - 6)(x + 3)$

$f(x) = x^2 - 3x - 18,$
$f'(x) = 2x - 3$

ii. $f(x) = (x - 2)(x - 1)$

$f(x) = x^2 - 3x + 2,$
$f'(x) = 2x - 3$

iii. $f(x) = (x + 2)\ (x + 7)$

$f(x) = x^2 + 9x + 14,$
$f'(x) = 2x + 9$

iv. $f(x) = (x - 6)^2$

$f(x) = (x - 6)(x - 6)$
$f(x) = x^2 - 12x + 36,$
$f'(x) = 2x - 12$

v. $f(x) = (3x - 4)(x - 1)$

$f(x) = 3x^2 - 7x + 4,$
$f'(x) = 6x - 7$

vi. $y = (10 + x)(2x + 3)$

$y = 2x2 + 23x + 30,$
$\frac{dy}{dx} = 4x + 23$

vii. $y = (3 - 4x)(4x + 2)$

$y = -16x^2 + 4x + 6,$
$\frac{dy}{dx} = -32x + 4$

viii. $y = (9x - 3)(2x - 1)$

$y = 18x^2 - 15x + 3,$
$\frac{dy}{dx} = 36x - 15$

ix. $f(x) = (2x + 4)^2$

5. Drawing Graphs of Functions and Gradient Functions

a. The gradient function $f'(x)$ of a quadratic will look like a:

straight Line

parabola

Straight line (The gradient function of a quadratic will be linear)

b. The gradient function $f'(x)$ of a cubic will look like a:

straight line

parabola

parabola (The gradient function of a cubic will be quadratic)

c. If the gradient function $f'(x)$ is linear, then the original function $f'(x)$ is:

linear

quadratic

cubic

quadratic

d. If the gradient function $f'(x)$ is quadratic, then the original function $f'(x)$ is:

linear quadratic cubic

cubic

e. If there is a turning point in the original function $f(x)$, what does that mean in the gradient function $f'(x)$?

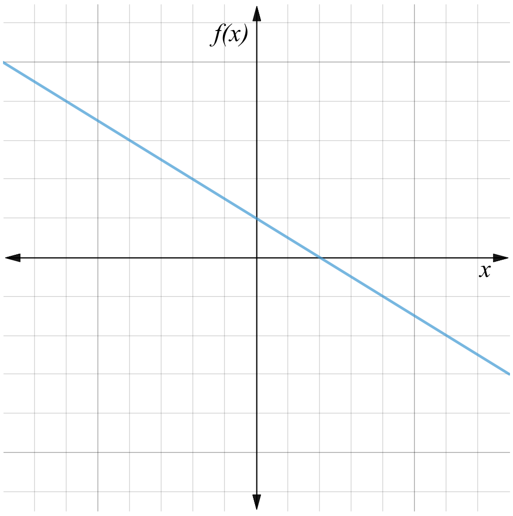
The gradient function will pass through the x–axis at every turning point.

f. The roots on a gradient function $f'(x)$ correspond to what in the original function?

The roots in a gradient function correspond to turning points in the original function.

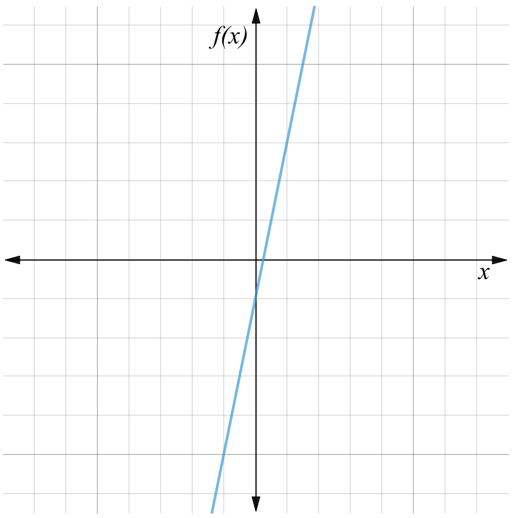
g. Identify which of the following graphs are positive functions, and which are negative functions:

i.



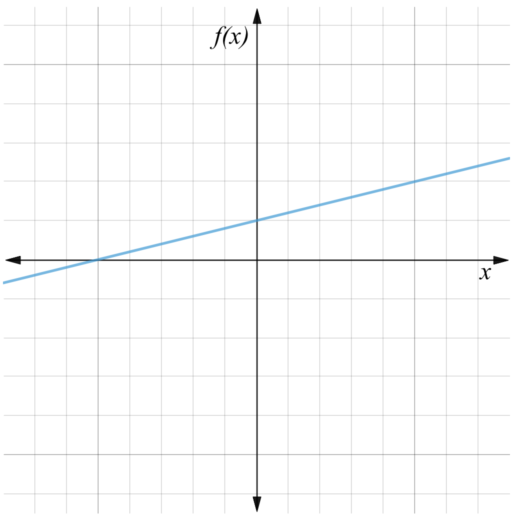
Negative linear function. This means the function has a negative x term in it (when fully expanded).

ii.



Positive linear function. This means the function has a positive x term in it (when fully expanded).

iii.



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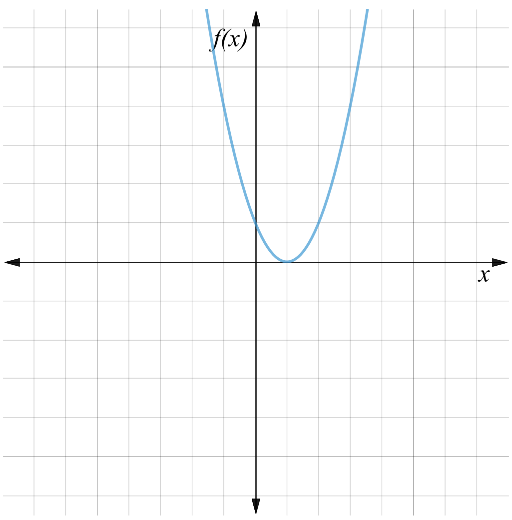
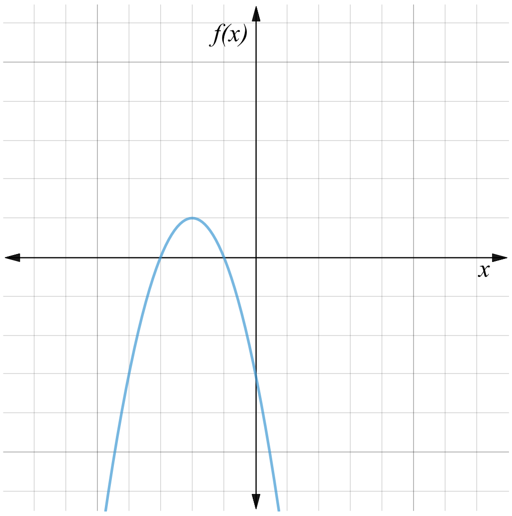


Positive linear function. This means the function has a positive x term in it (when fully expanded).

Positive quadratic function. This means the function has a positive x^2 term in it (when fully expanded).

v.



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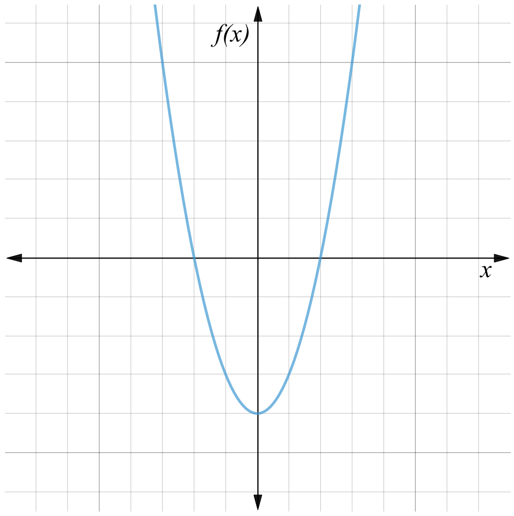
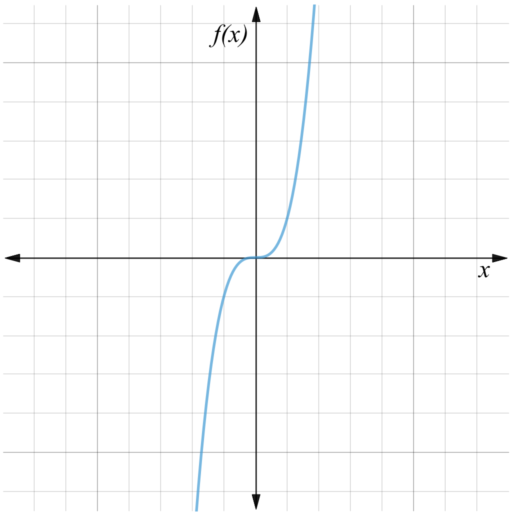


Negative quadratic function. This means the function has a negative x^2 term in it (when fully expanded).

Positive quadratic function. This means the function has a positive x2 term in it (when fully expanded).

vii.



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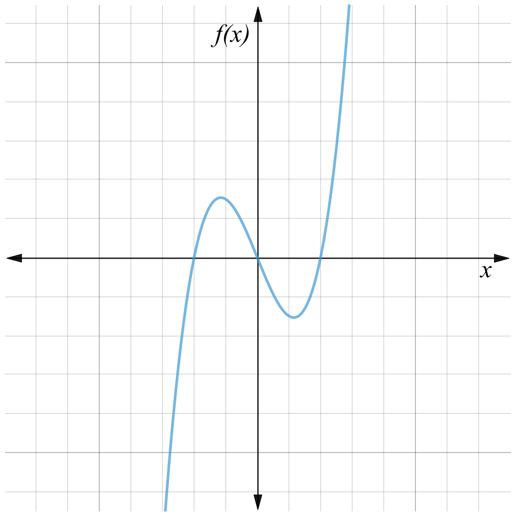
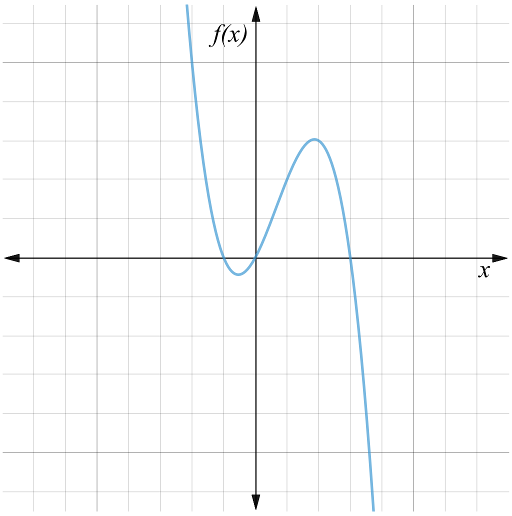


Positive cubic function. This means the function has a positive x^3 term in it (when fully expanded).

Positive cubic function. This means the function has a positive x^3 term in it (when fully expanded).

Answers

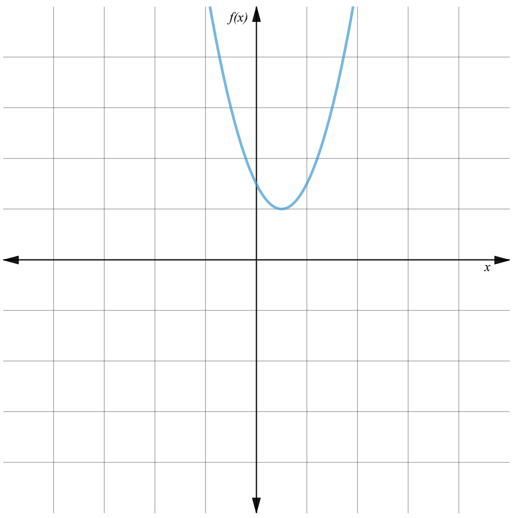
ix.

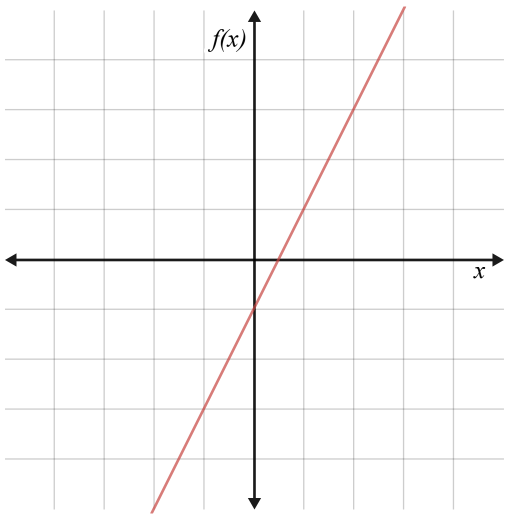


Negative cubic function. This means the function has a negative x^3 term in it (when fully expanded).

h. For the following graphs $f(x)$, draw the gradient function $f'(x)$ on the blank graph provided. All graphs have the same scale.

i.





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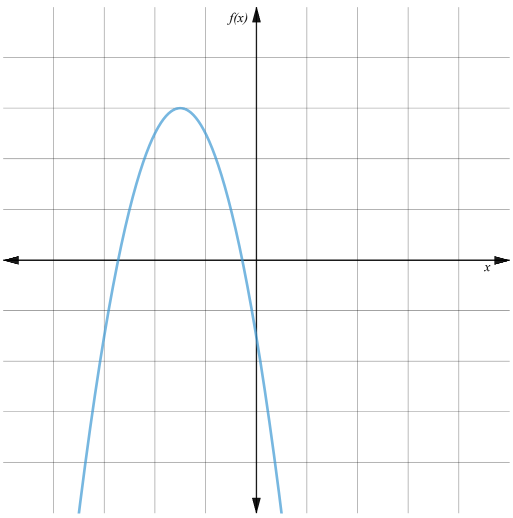

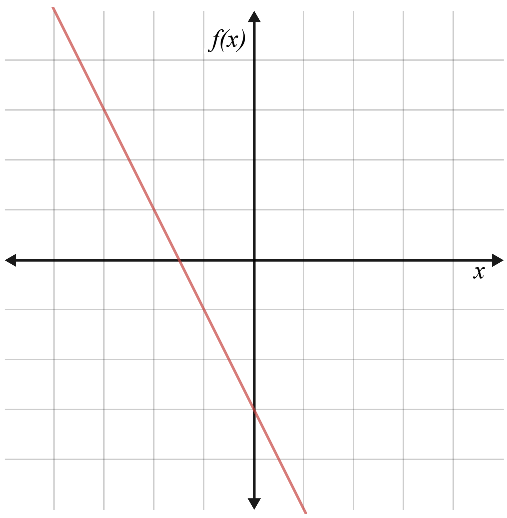

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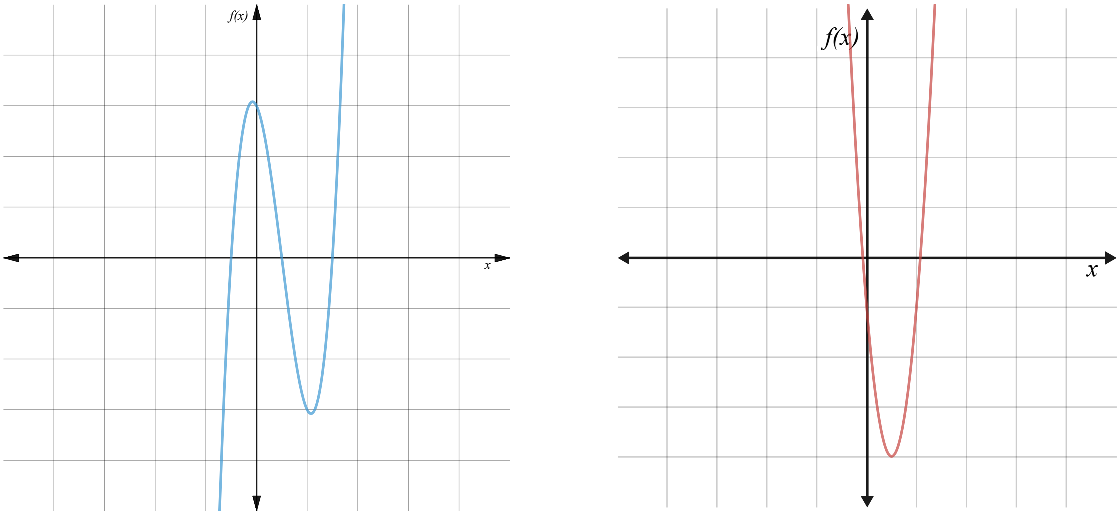

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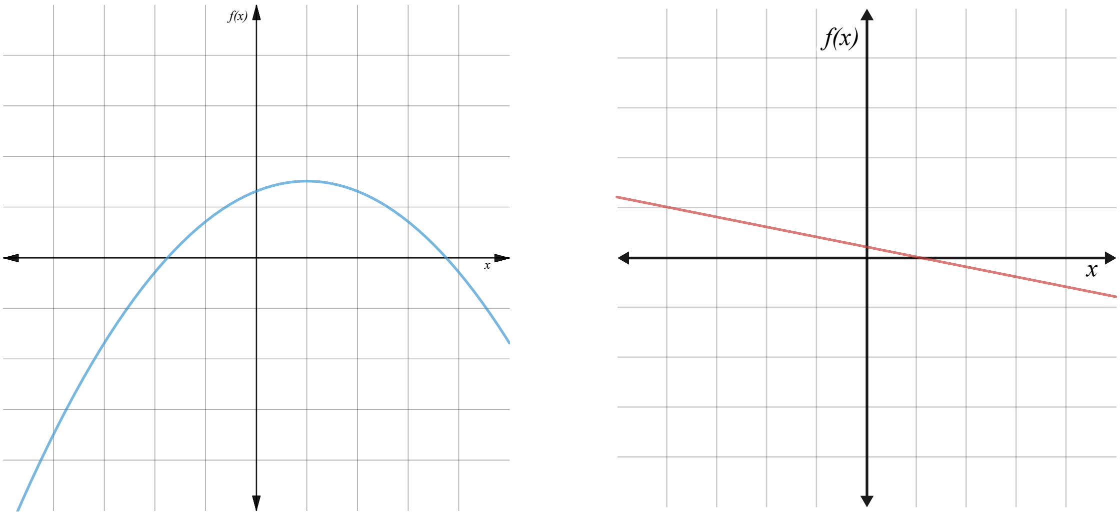

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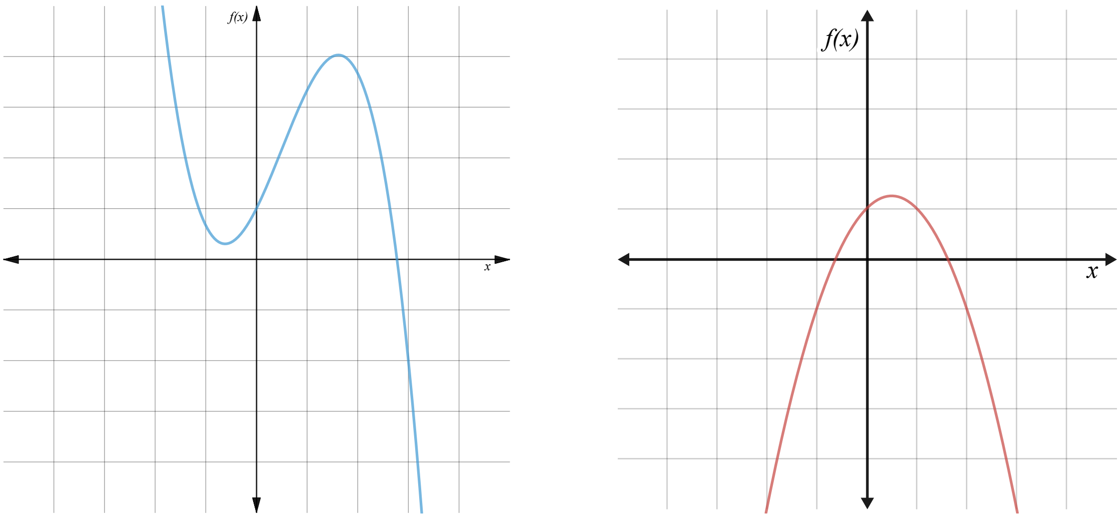

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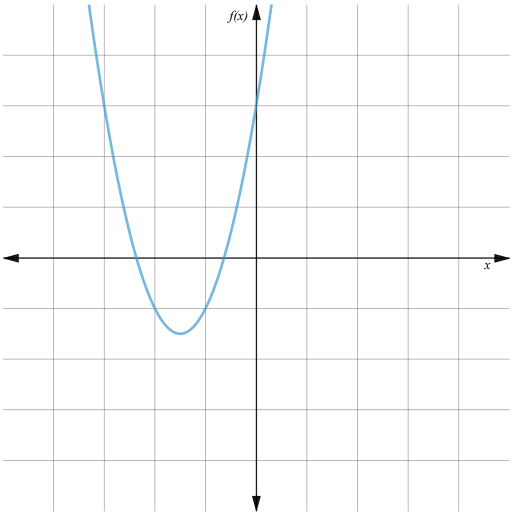
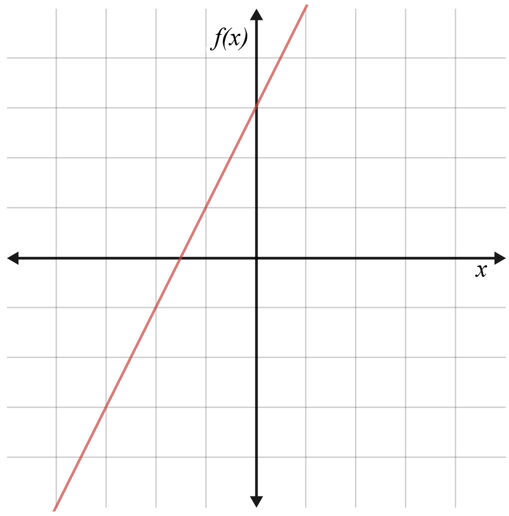
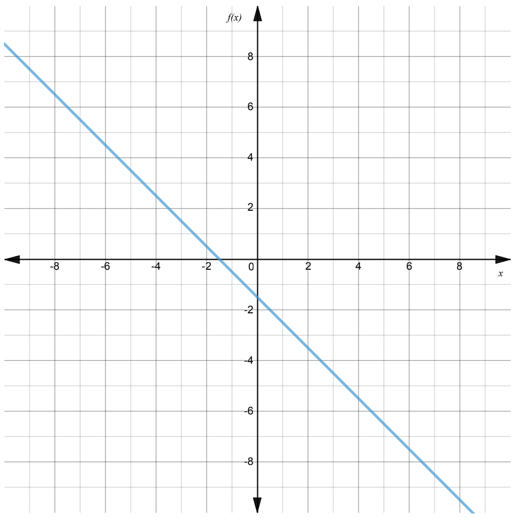
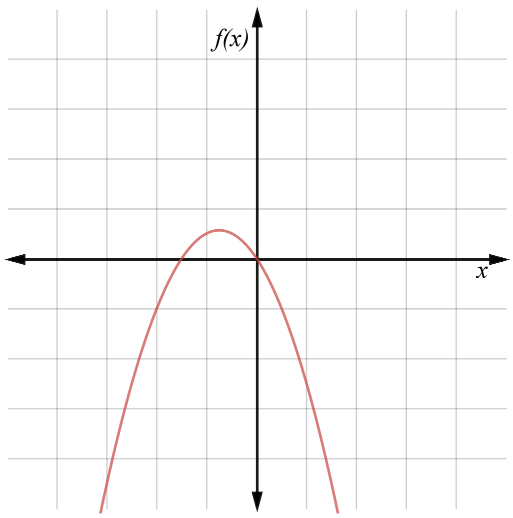




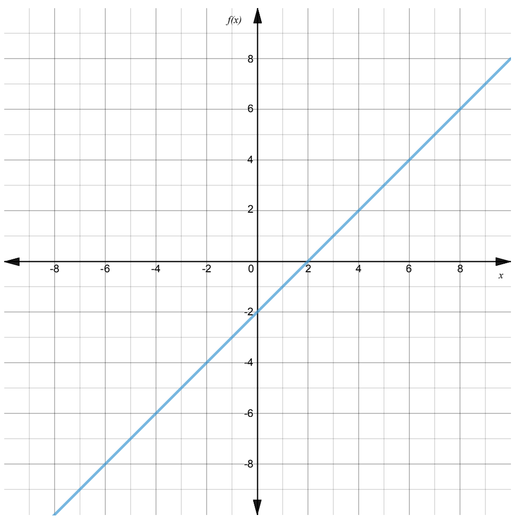
i. For the following graphs of gradient functions $f'(x)$, draw the original function $f(x)$ on the blank graph provided. All graphs have the same scale.

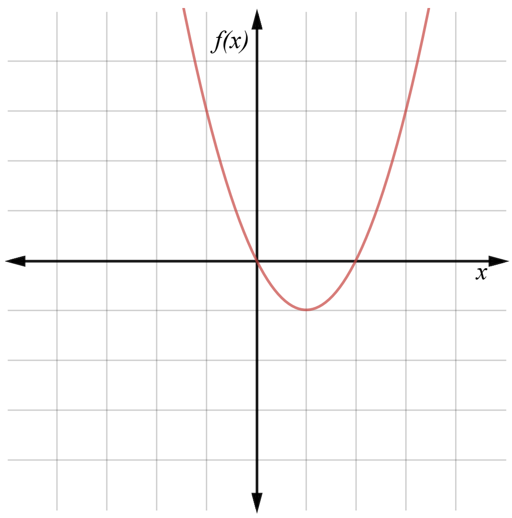
i. The graph of the original function passes through (0,0)



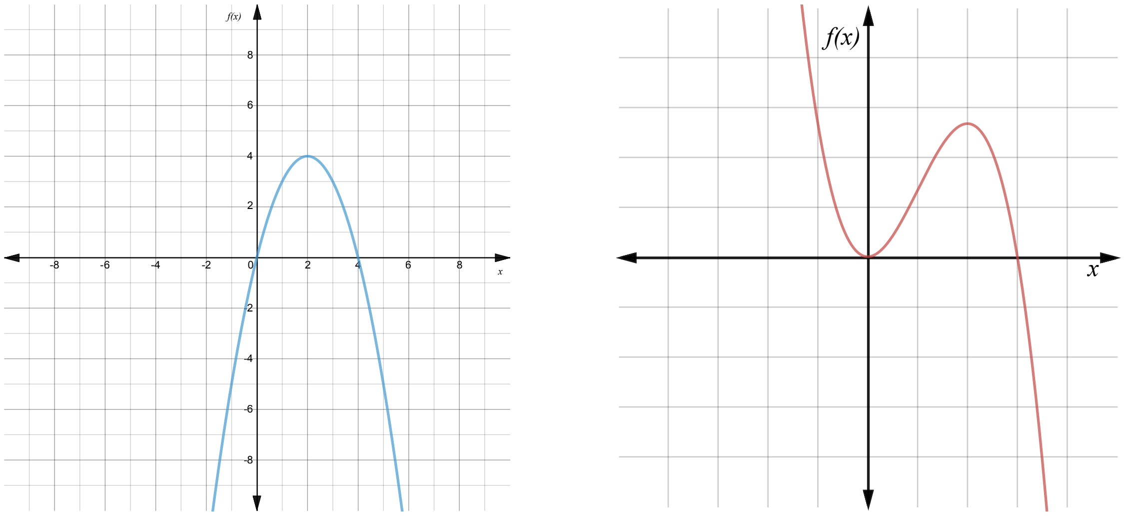


ii. The graph of the original function passes through (0,0)

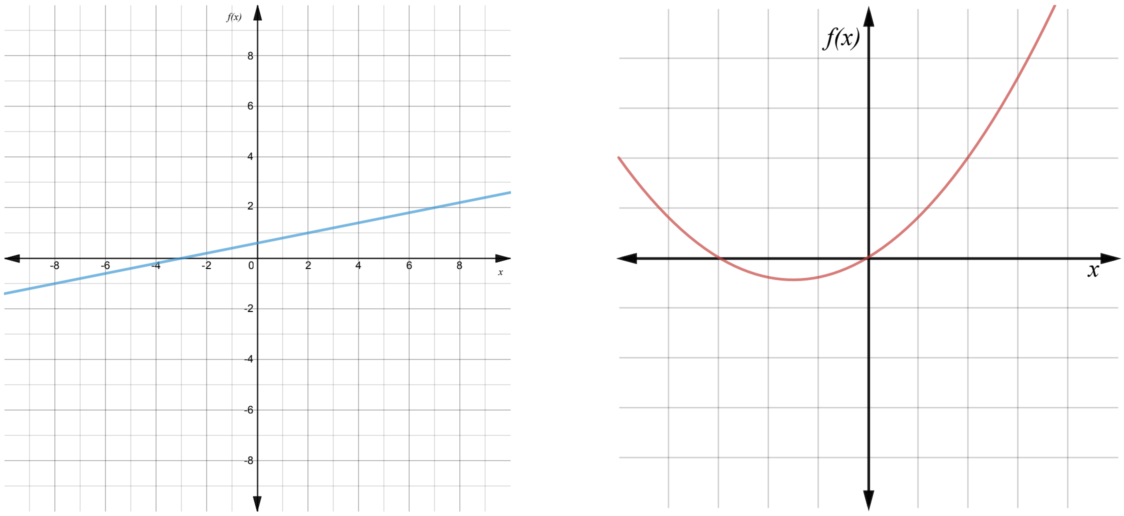




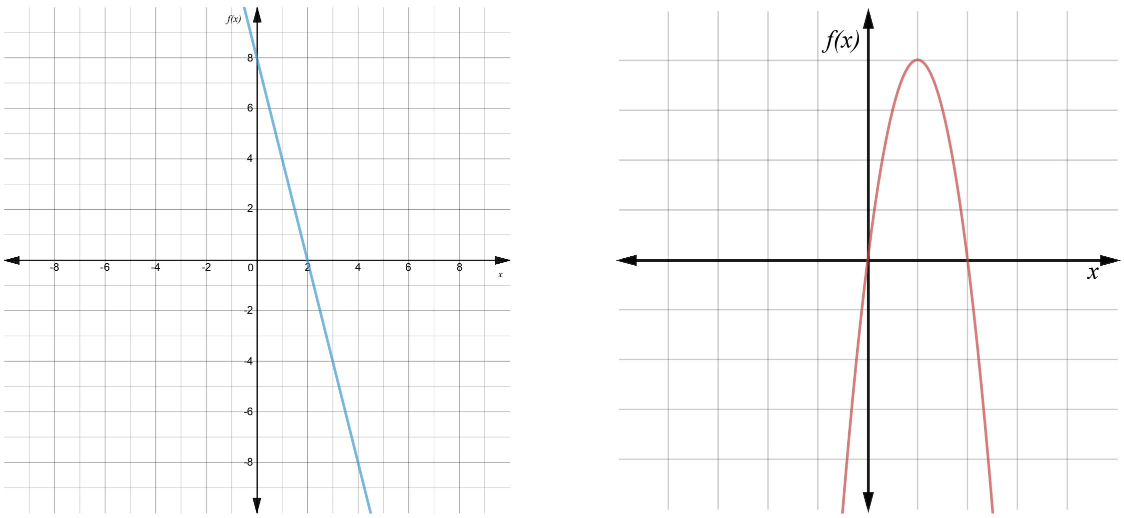
iii. The graph of the original function passes through (0,0)



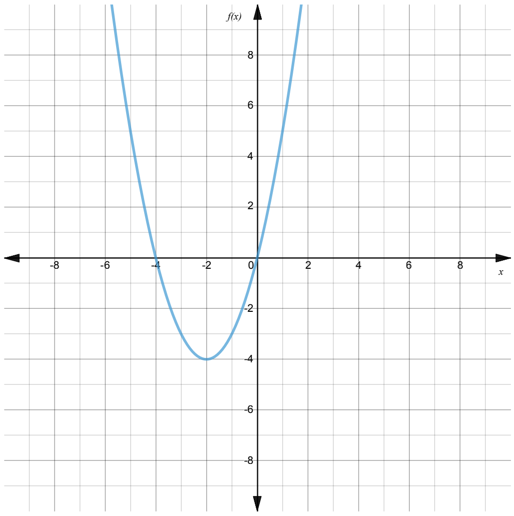
iv. The graph of the original function passes through (0,0)

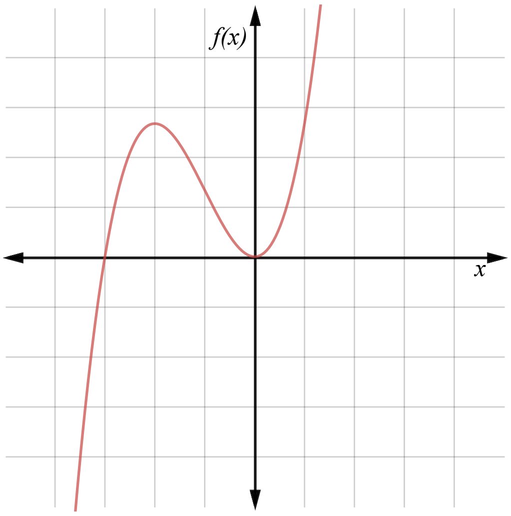


v. The graph of the original function passes through (0,0)



vi. The graph of the original function passes through (0,0)





6. Differentiating Interesting Powers

a. Differentiate

i. $y = x^{-2}$

$\frac{dy}{dx} = -\ 2x^{-3}$

ii. $y = 3x^{-1}$

$\frac{dy}{dx} = -\ 3x^{-2}$

iii. $y = 4x^{-3}$

$\frac{dy}{dx} = -12x^{-4}$

iv. $y = 5x^{-2}$

$\frac{dy}{dx} = -10x^{-3}$

v. $y = -6x^{-3}$

$\frac{dy}{dx} = 18x^{-4}$

vi. $y = -x^{-1}$

$\frac{dy}{dx} = x^{-2}$

vii. $f(x) = -3x^{-4}$

$f'(x) = 12x^{-5}$

viii. $f(x) = 5x^{-6}$

$f'(x) = -30x^{-7}$

ix. $f(x) = -12x^{-4}$

$f'(x) = 48x^{-5}$

x. $y = x^{0.5}$

$dydx = 0.5x^{-0.5}$

xi. $y = x^{\frac{2}{3}}$

$\frac{dy}{dx} = \frac{2}{3}\ x^{-\frac{1}{3}}$

xii. $y = 2x^{2.5}$

$\frac{dy}{dx} = 5x^{1.5}$

xiii. $f(x) = 3x^{\frac{1}{2}}$

$f'(x) = 1.5x^{-\frac{1}{2}}$

xiv. $f(x) = 4x^{\frac{3}{4}}$

$f'(x) = 3x^{-\frac{1}{4}}$

xv. $f(x) = -3x^{-\frac{1}{3}}$

$f'(x) = x^{-\frac{4}{3}}$

b. Differentiate

You will have to use fraction and power rules to simplify these expressions before you differentiate them. After simplifying the expressions, they can be differentiated the same way as in part a). For simplifying, remember that:

$\sqrt[a]{x} = x^{\frac{1}{a}}$ e.g. $\sqrt[3]{x} = x^{\frac{1}{3}}$ and $\sqrt{x} = x^{\frac{1}{2}}$

$\frac{1}{x^a} = x^{-a}$ e.g. $\frac{1}{x} = x^{-1}$ and $\frac{1}{x^4} = x^{-4}$

$(\frac{a}{b})^c = \frac{a^c}{b^c}$ e.g. $(\frac{2}{x})^2 = \frac{4}{x^2}$ and $(\frac{x^2}{2+x})^3 = \frac{x^6}{(2+x)^3}$

i. $y = \frac{4}{x^2}$

$y = 4x^{-2}$,

$\frac{dy}{dx} = -8x^{-3}$,

$\frac{dy}{dx} = -\frac{8}{x^3}$

ii. $y = \frac{3x}{6x^3}$

$y = \frac{1}{2x^2}$

$y = \frac{1}{2}x^{-2}$,

$\frac{dy}{dx} = -1x^{-3}$,

$\frac{dy}{dx} = -\frac{1}{x^3}$

iii. $y = \frac{2x}{(2x)^2}$

$y = \frac{1}{2x}$,

$y = \frac{1}{2}x^{-1}$,

$\frac{dy}{dx} = -\frac{1}{2}x^{-2}$,

$\frac{dy}{dx} = -\frac{1}{2x^2}$

iv. $f(x) = 4x^6$

$f(x) = (4x^6)^{\frac{1}{2}}$,

$f(x) = 2x^3$,

$f'(x) = 6x^2$

v. $f(x) = \sqrt{\frac{16}{x}}$

$f(x) = (\frac{16}{x})^{\frac{1}{2}}$,

$f(x) = \frac{4}{x^{\frac{1}{2}}}$,

$f(x) = 4x^{-\frac{1}{2}}$,

$f'(x) = -2x^{-\frac{3}{2}}$,

$f'(x) = \frac{-2}{x^{\frac{3}{2}}}$

vi. $y = \sqrt{\frac{16}{4x^2}}$

$y = \frac{4}{2x}$,

$y = 2x^{-1}$,

$\frac{dy}{dx} = -2x^{-2}$

vii. $y = \sqrt{\frac{3x}{5x^6}}$

$y = \frac{3}{5}x^{-5}$

$\frac{dy}{dx} = -3x^{-6}$

viii. $f(x) = \sqrt{5x^5}$

$f(x) = (5x^5)^{\frac{1}{2}}$

$f(x) = \sqrt{5}x^{\frac{5}{2}}$

$f'(x) = \frac{5\sqrt{5}}{2}x^{\frac{3}{2}}$

ix. $f(x) = \sqrt{4x^8}$

$f(x) = (4x^8)^{\frac{1}{2}}$

$f(x) = 2x^4$

$f'(x) = 8x^3$

7. Calculating the gradient at a point

Find the gradient of the following functions, at the point given.

First differentiate to find the gradient function (some functions will need to be simplified or expanded first). Then substitute the x–value into the gradient equation to find the gradient at that point.

a. $y = 3x^2 + 5x$ at $x = 1$

$\frac{dy}{dx} = 6x + 5$

Substitute $x = 1$

$\frac{dy}{dx} = 6(1) + 5$

$\frac{dy}{dx} = 11$

The gradient at $x = 1$ is 11.

b. $f(x) = 5(x + 7)$ at $x = -2$

$f(x) = 5x + 35$

$f'(x) = 5$

$f'(-2) = 5$

(Gradient is always 5, regardless of the x value)

c. $y = 2x^2 - 3x + 4$ at $x = 2$

$\frac{dy}{dx} = 4x - 3$

Substitute $x = 2$

$\frac{dy}{dx} = 4(2) - 3$

$\frac{dy}{dx} = 5$

The gradient at $x = 2$ is 5

d. $y = 5x^3 - 3x^2 + 4x + 900$ at $x = 0.5$

$\frac{dy}{dx} = 15x^2 - 6x + 4$

Substitute $x = 0.5$

$\frac{dy}{dx} = 15(0.5)^2 - 6(0.5) + 4$

$\frac{dy}{dx} = 4.75$

The gradient at $x = 0.5$ is 4.75

e. $y = -4x^4 - 16x^{-3} + \frac{4}{x}$ at $x = 2$

$y = -4x^4 - 16x^{-3} + 4x^{-1}$

$\frac{dy}{dx} = -16x^3 + 48x^{-4} - 4x^{-2}$

Substitute $x = 2$

$\frac{dy}{dx} = -16(2)^3 + 48(2)^{-4} - 4(2)^{-2}$

$\frac{dy}{dx} = -126$

The gradient at $x = 2$ is -126

f. $f(x) = (x-3)(x-6)$ at $x = 5$

$f(x) = x^2 - 9x + 18$

$f'(x) = 2x - 9$

Substitute $x = 5$

$f'(5) = 2(5) - 9$

$f'(5) = 1$

The gradient at $x = 5$ is 1

g. $f(x) = (x-6)^2$at $x = 2$

$f(x) = x^2 - 12x + 36$
$f'(x) = 2x - 12$
Substitute $x = 2$
$f'(2) = 2(2) - 12$
$f'(x)=-8$
The gradient at $x = 2$ is -8

h. $f(x) = (x + 1)(2x-3)$ at $x = -1$

$f(x) = 2x^2 - x - 3$
$f'(x) = 4x - 1$
Substitute $x = -1$
$f'(-1) = 4(-1) - 1$
$f'(-1) = -5$
The gradient at $x = -1$ is -5

i. $y = (2x + 5)(3x - 6)$ at $x = 0$

$y = 6x^2 + 3x - 30$
$\frac{dy}{dx} = 12x + 3$
Substitute $x = 0$
$\frac{dy}{dx} = 12(0) + 3$
$\frac{dy}{dx} = 3$
The gradient at $x = 0$ is 3

j. $y = (-3 + 2)(x)$ at $x = 3$

$y = -1x$
$\frac{dy}{dx} = -1$
(Gradient is always -1, regardless of the x value)

k. $y = (2x^2 - 4)(x + 3)$ at $x = 1$

$y = 2x^3 + 6x^2 - 4x - 12$
$dydx = 6x^2 + 12x - 4$
Substitute $x = 1$
$\frac{dy}{dx} =6(1)^2 + 12(1) - 4$
$\frac{dy}{dx} = 14$
The gradient at $x = 1$ is 14

l. $f(x) = \frac{500}{2^x}$ at $x = 5$

$f(x) = 500x^{-2}$
$f'(x) = -1000x^{-3}$
Substitute $x = 5$
$f'(5) = -1000(5)^{-3}$
$f'(5) = -8$
The gradient at $x = 5$ is -8

m. $y = \sqrt{16x^{-3}}$ at $x = 2$

$y = 4x^{-\frac{3}{2}}$
$\frac{dy}{dx} = -6x^{-\frac{5}{2}}$
Substitute $x = 2$
$\frac{dy}{dx} = -6(2)^{-\frac{5}{2}}$
$\frac{dy}{dx} = -1.061(3dp)$
The gradient at $x = 2$ is -1.061

n. $y = \sqrt{\frac{4}{x^6}}$ at $x = 1$

$y = \frac{2}{x^3}$
$y = 2x^{-3}$
$\frac{dy}{dx} = -6x^{-4}$
Substitute $x = 1$
$\frac{dy}{dx} = -6(1)^{-4}$
$\frac{dy}{dx} = -6$
The gradient at $x = 1$ is -6

o. $f(x) = \sqrt{\frac{x^4}{4}}$ at $x = 0.2$

$f(x) = \frac{x^2}{2}$
$f'(x) = x$
Substitute $x = 0.2$
$f'(0.2) = 0.2$
The gradient at $x = 0.2$ is 0.2

p. $f(x) = \frac{5}{x^4} - 10x + 12x^3$ at $x = 10$

8. The Tangent Line

a. What does a tangent line have in common with the curve it's drawn on?

1. The gradient of the tangent is the same as the curve the the point where they touch
2. Both lines pass through the same point (where the tangent line is 'tangent' to the curve)

b. What are the steps to determine the equation of a tangent for a curve at a point x?

1. Determine the function's value (y) at point x, by substituting x into the equation given.
2. Differentiate the function
3. Determine the value of the gradient at point x by substituting x into the differentiated equation
4. Determine the constant c in the tangent line equation $y = mx + c$ by substituting the coordinates of the point into the equation.

c. Take the function $f(x) = 7x^3 - 4x + 121$

i. What are the coordinates at the point on the curve when $x = 4$?

$f(4) = 7(4)^3 - 4(4) + 121$
$f(4) = 553$
The coordinates when $x=4$ are $(4, 553)$.

ii. What is the equation of the gradient of the curve?

Equation of the gradient requires differentiating
$f'(x) = 21x^2 - 4$

iii. What is the gradient of the tangent to the curve at $x = 4$?

$f'(4) = 21(4)^2 - 4$
$f'(4) = 332$
The gradient of the tangent to the curve at $x = 4$ is 332.

iv. What is the equation of the tangent line at $x = 4$?

$y = 332x + c$
Sub in the coordinates that we know (4, 553) and rearrange to find c.
$553 = 332(4) + c$
$c = -775$
The equation of the tangent at $x = 4$ is:
$y = 332x - 775$

d. For $f(x) = (x - 2)(x + 3)$

i. What are the coordinates at the point on the curve when $x = 1$?

$f(1) = (1 - 2)(1 + 3)$
$f(1) = -4$
The coordinates when $x=1$ are $(1, -4)$.

ii. What is the equation of the gradient of the curve?

Equation of the gradient requires differentiating, but first we need to expand the original expression
$f(x) = x^2 + x - 6$
$f'(x) = 2x + 1$

iii. What is the gradient of the tangent to the curve at $x = 1$?

$f'(1) = 2(1) + 1$
$f'(1) = 3$

iv. What is the equation of the tangent line at $x = 1$?

$y = 3x + c$
Sub in the co–ordinates $(1,-4)$ and rearrange for c
$-4 = 3(1) + c$
$c = -7$
$y = 3x - 7$

e. For $f(x) = 10x - 3x^2 + 15$

i. What are the coordinates at the point on the curve when $x = -3$?

$f(-3) = 10(-3)-3(-3)^2 + 15$
$f(-3) = -42$
The coordinates when $x = -3$ are $(-3, -42)$.

ii. What is the equation of the gradient of the curve?

Equation of the gradient requires differentiating
$f'(x) = 10 - 6x$

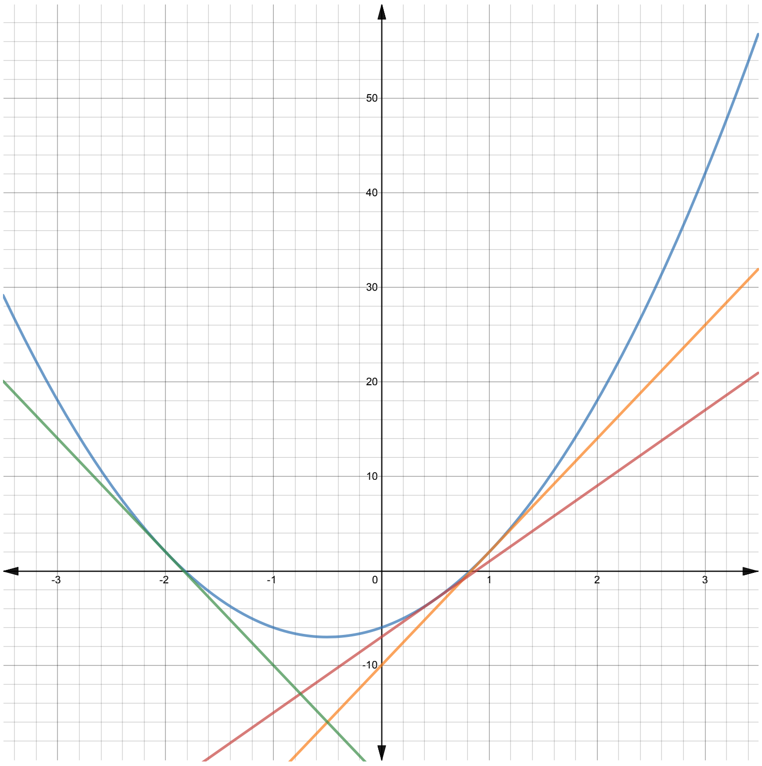
iii. What is the gradient of the tangent to the curve at $x = -3$?

$f'(-3) = 10 - 6(-3)$
$f'(-3) = 28$

iv. What is the equation of the tangent line at $x = -3$?

$y = 28x + c$
Sub in the co–ordinates $(-3,-42)$ and rearrange for c
$-42 = 28(-3) + c$
$c = 42$
$y = 28x + 42$

f. This graph shows the equation $y = 4x^2 + 4x - 6$ with tangent lines drawn at the points $x = -2$, $x = 0.5$, $x = 1$



i. Estimate the equations of the tangent lines from the graph shown:

Using the general equation for a straight line $y = mx + c$, we know c is the y intercept, which is easy to see on the graph.

For the gradient, m, we could use rise over run to estimate the value. The red line doesn't pass through any obvious exact points on the graph, so rough values would probably have to be used.

For the answers, check next section

1. At $x = -2$ (the green line)

2. At $x = 0.5$ (the red line)

3. At $x = 1$ (the orange line)

ii. Calculate the equation of the tangent line at the point where $x = -2$

At $x = -2$,
$y = 4(-2)^2 + 4(-2) - 6$
$y = 2$
Differentiate the parabola equation to get:
$\frac{dy}{dx} = 8x + 4$
At $x = -2$, $\frac{dy}{dx} = -12$
Now we start to make our tangent equation, using $y=mx+c$.
$y = -12x + c$
$2 = -12(-2) + c$
$c = -22$
The equation of the tangent line is:
$y = -12x - 22$

iii. Calculate the equation of the tangent line at the point where $x = 0.5$

At $x = 0.5$,
$y = 4(0.5)^2 + 4(0.5) - 6$
$y = -3$
Differentiate the parabola equation to get:
$\frac{dy}{dx} = 8x + 4$
At $x = 0.5$, $\frac{dy}{dx} = 8$
Now we start to make our tangent equation.
$y = 8x + c$
$-3 = 8(0.5) + c$
$c = -7$
The equation of the tangent line is:
$y = 8x - 7$

iv. Calculate the equation of the tangent line at the point where $x = 1$

At $x = 1$,
$y = 4(1)^2 + 4(1) - 6$
$y = 2$
Differentiate the parabola equation to get:
$\frac{dy}{dx} = 8x + 4$
At $x = 1$, $\frac{dy}{dx} = 12$
Now we start to make our tangent equation.
$y = 12x + c$
$2 = 12(1) + c$
$c = -10$
The equation of the tangent line is:
$y = 12x - 10$

g. Calculate the equation of the tangent line to $f(x) = (x{-}6)(x - 3)$at $x = 1.5$

$f(x) = x^2 - 9x + 18$
$f(1.5) = 6.75$
At $x = 1.5,\ y = 6.75$
Differentiate to get:
$f'(x) = 2x - 9$
$f'(1.5) = -6$
The gradient of the tangent at $x = 1.5$is -6, which we can use in our tangent equation.
$y = -6x + c$
$6.75 = -6(1.5) + c$
$c = 15.75$
The equation of the tangent line is:
$y = -6x + 15.75$

h. Calculate the equation of the tangent line to $f(x) = 3x^2 + 8x - 2$at $x = 0$

$f(0) = 3(0)2 + 8(0) - 2$
$f(0) = -2$
At $x = 0,\ y = -2.$
Differentiate to get:
$f'(x) = 6x + 8$
$f'(0) = 8$
At $x = 0$, the gradient is 8.
$y = 8x + c$
$-2 = 8(0) + c$
$c = -2$
The equation of the tangent is:
$y = 8x - 2$

i. Calculate the equation of the tangent line to $f(x) = 2x^4 - 11x^2 + 7$at $x = 4$

$f(4) = 2(4)^4 - 11(4)^2 + 7$
$f(4) = 343$
At $x = 4,\ y = 343.$
Differentiate to get
$f'(x) = 8x^3 - 22x$
$f'(4) = 424$
At $x = 4$, the gradient is 424.
$y = 424x + c$
$343 = 424(4) + c$
$c = -1353$
The equation of the tangent is:
$y = 424x - 1353$

9. Turning Points

a. What is the gradient at a turning point?

Zero.

b. How do you find the coordinates of a turning point?

Differentiate the equation.
Make the differentiated equation equal to zero and solve for x.
Once you have x, substitute x into the original equation to find the y value.

c. Find the x coordinates of the turning points for these equations:

i. $y = 5x^3 - 6$

$\frac{dy}{dx} = 15x^2 = 0$
$x = 0$

ii. $f(x) = 12x^2 - 6x + 11$

$f'(x) = 24x - 6 = 0$
$x = 0.25 \text{ or } \frac{1}{4}$ (or equivalent)

iii. $y = x^2 + 2x + 4$

$\frac{dy}{dx} = 2x + 2 = 0$
$x = -1$

iv. $f(x) = 6x^2 + 9x + 12$

$f'(x) = 12x + 9 = 0$
$x = -\ 0.75 \text{ or } \frac{3}{4}$ (or equivalent)

v. $y = 5x^2 + 6x + 3$

$\frac{dy}{dx} = 10x + 6 = 0$
$x = -\ 0.6 \text{ or } \frac{6}{10}$ (or equivalent)

vi. $f(x) = 9x^2 - 90x + 430$

$f'(x) = 18x - 90 = 0$
$x = 5$

vii. $y = x^3 - 10x^2 + 5x + 210$

$\frac{dy}{dx} = 3x^2 - 20x + 5 = 0$
Solved with quadratic formula or graphics calculator solver function.
$x = 6.41 \text{ and } x = 0.26$

viii. $y = 4x^3 + 3x^2 - 6x$

$\frac{dy}{dx} = 12x^2 + 6x - 6 = 0$
Solved with quadratic formula, factorising, or graphics calculator solver function.
$x = 0.5 \text{ (or } \frac{1}{2}) \text{ and } x = -\ 1$

ix. $f(x) = 3x^3 - 9x + 11$

$f'(x) = 9x^2 - 9 = 0$
Solved with quadratic formula, factorising, or graphics calculator solver function.
$x = 1 \text{ and } x = -\ 1$

x. $f(x) = 6x3 - 2x^2 - 18x + 45$

$f'(x) = 18x^2 - 4x - 18 = 0$
Solved with quadratic formula or graphics calculator solver function.
$x{=}1.117 \text{ and } x = -\ 0.895$

d. Determine the coordinates of the turning points for the following questions:

i. $f(x) = 12x^2 - 144x + 14$

$f'(x) = 24x - 144 = 0$
$x = 6$
$f(6) = -\ 418$
Turning point is $(6, -418)$

ii. $y = 2x^2 - 14x + 13$

$\frac{dy}{dx} = 4x - 14 = 0$
$x = 3.5$
$f(3.5) = -\ 11.5$
Turning point is $(3.5, -11.5)$

Answers

iii. $f(x) = -6x^2 + 2x + 1$

$f'(x) = -12x + 2 = 0$
$x = \frac{1}{6}$
$f(\frac{1}{6}) = \frac{7}{6}$ (or 1.1667)
Turning point is $(\frac{1}{6}, \frac{7}{6})$

iv. $y = 140x - 1500$

$\frac{dy}{dx} = 140 \neq 0$
This cannot equal zero therefore this graph has no turning points.

v. $f(x) = 6x^3 - 4x^2 + 16$

$f'(x) = 18x^2 - 8x = 0$
$x = \frac{4}{9}$ and $x = 0$
$f(\frac{4}{9}) = 15.74$
$f(0) = 16$
Turning points are $(\frac{4}{9}, 15.74)$ and $(0, 16)$

vi. $y = 3x^3 + 10x^2 + x + 60$

$\frac{dy}{dx} = 9x^2 + 20x + 1 = 0$
$x = -0.05$ and $x = -2.17$
When $x = -0.05, y = 59.97$
When $x = -2.17, y = 74.26$
Turning points are $(-0.05, 59.97)$ and $(-2.17, 74.26)$

vii. $f(x) = 10x^3 + 8x^2 - 11x + 7$

$f'(x) = 30x^2 + 16x - 11 = 0$
$x = 0.39$ and $x = -0.93$
$f(0.39) = 4.52$
$f(-0.93) = 16.11$
Turning points are $(0.39, 4.52)$ and $(-0.93, 16.11)$

viii. $y = x^3 - x^2 - 10x + 1000$

$\frac{dy}{dx} = 3x^2 - 2x - 10 = 0$
$x = 2.19$ and $x = -1.52$
When $x = 2.19, y = 983.81$
When $x = -1.52, y = 1009.38$
Turning points are $(2.19, 983.81)$ and $(-1.52, 1009.38)$

Answers

e. How do you determine if a turning point is a maximum or minimum?

Take the original function, and substitute in x values which are higher and lower than the turning point. If the y/function values are higher than the turning point, then it is a minimum. If the y/function values either side of the turning point are lower, then the turning point is a maximum.

f. Determine the coordinates of the turning points for the following equations and determine if they are maximums or minimums:

i. $f(x) = 3x^2 - 9x - 17$

$f'(x) = 6x - 9 = 0$
$x = 1.5$
$f(1.5) = -23.75$
Turning point is $(1.5, -23.75)$
Try $x = 0, f(0) = -17$
Try $x = 2, f(2) = -23$
As the value of the function on either side of the turning point is higher than -23.75, the turning point must be a minimum.

ii. $y = x^2 - 14x + 130$

$\frac{dy}{dx} = 2x - 14 = 0$
$x=7$
When $x=7, y=81$
Turning point is $(7, 81)$
Try $x = 0, y = 130$
Try $x = 10, y = 90$
As the value of y on either side of the turning point is higher than 81, the turning point must be a minimum.

iii. $f(x) = -2x^2 + 5x - 10$

$f'(x) = -4x + 5 = 0$
$x = 1.25$
$f(1.25) = -6.875$
Turning point is $(1.25, -6.875)$
Try $x = 0, f(0) = -10$
Try $x = 3, f(3) = -13$
As the value of the function on either side of the turning point is lower than -6.875, the turning point must be a maximum

iv. $y = 0.5x^2 - x - 13.5$

$\frac{dy}{dx} = x - 1 = 0$
$x=1$
When $x = 1, y = -14$
Turning point is $(1, -14)$
Try $x = 0, y = -13.5$
Try $x = 2, y = -13.5$
As the value of the function on either side of the turning point is higher than -14, the turning point must be a minimum.

10. Integration (Anti-differentiation)

a. What is integration?

Integration is the act of finding the equation of a function if we have the equation of the gradient.

b. How does integration relate to differentiation?

It is the opposite of differentiation.

c. What are the steps required to integrate a basic function?

1. Add one to the power
2. Divide the term by the new power
3. Add +C to the end.

d. What is the symbol which represents integration?

$\int ... dx$

e. Why do we add +C when integrating?

We add +C to represent the term which is lost when differentiating. For example, if $y = 2x + 7$ then $\frac{dy}{dx} = 2$. But if we integrate $\frac{dy}{dx} = 2$ we get $2x$, so we add +C to represent the 7 which gets lost.

f. Integrate the following:

i. $\int x \, dx$

$\frac{1}{2}x^2 + c$

ii. $\int x^6 \, dx$

$\frac{1}{7}x^7 + c$

iii. $\int x^{11} \, dx$

$\frac{1}{12}x^{12} + c$

iv. $\int x^2 \, dx$

$\frac{1}{3}x^3 + c$

Answers

v. $\int x^4 \, dx$

$\frac{1}{5}x^5 + c$

vi. $\int x^8 \, dx$

$\frac{1}{9}x^9 + c$

vii. $\int x^9 \, dx$

$\frac{1}{10}x^{10} + c$

viii. $\int x^{50} \, dx$

$\frac{1}{51}x^{51} + c$

ix. $\int x^{66} \, dx$

$\frac{1}{67}x^{67} + c$

g. Integrate the following and simplify (where possible)

i. $\int 6x^2 \, dx$

$\frac{6}{3}x^3 + c$
$2x^3 + c$

ii. $\int 14x6 \, dx$

$\frac{14}{7}x^7 + c$
$2x^7 + c$

iii. $\int 3x \, dx$

$\frac{3}{2}x^2 + c$

iv. $\int 63 \, dx$

$63x + c$

v. $\int 18x^5 \, dx$

$\frac{18}{6}x^6 + c$
$3x^6 + c$

vi. $\int 11x^3 \, dx$

$\frac{11}{4}x^4 + c$

vii. $\int 13x^{19} \, dx$

$\frac{13}{20}x^{20} + c$

viii. $\int 2x^6 \, dx$

$\frac{2}{7}x^7 + c$

ix. $\int 1.5x^2 \, dx$

$\frac{1.5}{3}x^3 + c$
$0.5x^3 + c$

x. $\int 24x^5 \, dx$

$\frac{24}{6}x^6 + c$
$4x^6 + c$

xi. $\int 42x^6 \, dx$

$\frac{42}{7}x7 + c$
$6x^7 + c$

xii. $\int 100x^9 \, dx$

$\frac{100}{10}x^{10} + c$
$10x^{10} + c$

xiii. $\int 100 \, dx$

$100x + c$

xiv. $\int 20x \, dx$

$\frac{20}{2}x^2 + c$
$10x^2 + c$

Answers

xv. $\int 15x^6\ dx$

$\frac{15}{7}x^7 + c$

h. Integrate the following and simplify (where possible): (Hint: when you're given $f'(x)$, you must integrate to find $f(x)$)

i. $\int 20x + 62 + 12x^3\ dx$

$\frac{20}{2}x^2 + 62x + \frac{12}{4}x^4 + c$

$10x^2 + 62x + 3x^4 + c$

ii. $\int 15x^4 + 19x^5\ dx$

$\frac{15}{5}x^5 + \frac{19}{6}x^6 + c$

$3x^5 + \frac{19}{6}x^6 + c$

iii. $f'(x) = 12x^5 + 3x^9$

$f(x) = \int 12x^5 + 3x^9\ dx$

$f(x) = \frac{12}{6}x^6 + \frac{3}{10}x^{10} + c$

$f(x) = 2x^6 + \frac{3}{10}x^{10} + c$

iv. $f'(x) = 6x^8 + 2x + 17$

$f(x) = \int 6x^8 + 2x + 17\ dx$

$f(x) = \frac{6}{9}x^9 + 22x^2 + 17x + c$

$f(x) = \frac{2}{3}x^9 + x^2 + 17x + c$

v. $\int 12x^{11} + 3x^5 + 4x\ dx$

$\frac{12}{12}x^{12} + \frac{3}{6}x^6 + \frac{4}{2}x^2 + c$

$x^{12} + 0.5x^6 + 2x^2 + c$

vi. $\int 1 + 2x + 4x^5\ dx$

$x + \frac{2}{2}x^2 + \frac{4}{6}x^6 + c$

$x + x^2 + \frac{2}{3}x^6 + c$

vii. $f'(x) = 4x^4 + 3x^3$

$f(x)=\int 4x^4 + 3x^3\ \ dx$

$f(x) = \frac{4}{5}x^5 + \frac{3}{4}x^4 + c$

viii. $f'(x) = 5x^4 + 7x^9 + 18$

$f(x) = \int 5x^4 + 7x^9 + 18\ dx$

$f(x) = \frac{5}{5}x^5 + \frac{7}{10}x^{10} + 18x + c$

$f(x) = x^5 + \frac{7}{10}x10 + 18x + c$

ix. $f'(x) = 18x2 + 100 + 16x^7$

$f(x) = \int 18x^2 + 100 + 16x^7\ dx$

$f(x) = \frac{18}{3}x^3 + 100x + \frac{16}{8}x^8 + c$

$f(x) = 6x^3 + 100x + 2x^8 + c$

i. Integrate the following and simplify: (Hint: You might have to expand/simplify before integrating)

i. $f'(x) = 3x(x^5 + 1)\ dx$

$f'(x) = 3x^6 + 3x$

$f(x) = \frac{3}{7}x^7 + \frac{3}{2}x^2 + c$

ii. $\int 2x^7 + 12 + 3x^7 + 40\ dx$

$\frac{2}{8}x8 + 12x + \frac{3}{8}x8 + 40x + c$

$\frac{5}{8}x8 + 52x + c$

iii. $\int (2x - 1)(3x^2 + 3)\ dx$

$\int 6x^3 - 3x^2 + 6x - 3\ dx$

$\frac{6}{4}x4 - x^3 + 3x^2 - 3x + c$

iv. $f'(x) = (x + 2)^2$

$f'(x) = x^2 + 4x + 4$

$f(x) = \frac{1}{3}x3 + 2x^2 + 4x + c$

v. $\int \frac{12x^{11} + 3x^5 + 4x}{2x} dx$

$\int 6x^{10} + 1.5x^4 + 2 \, dx$

$\frac{6}{11}x^{11} + \frac{1.5}{5}x^5 + 2x + c$

vii. $\int \frac{x^2 + 4x - 5}{x - 1} dx$

$\int \frac{(x - 1)(x - 5)}{x - 1} dx$

$\int x - 5 \, dx$

$\frac{1}{2}x^2 - 5x + c$

vii. $f'(x) = x^5(12 + x^2)$

$f'(x) = 12x^5 + x^7$

$f(x) = \frac{12}{6}x^6 + \frac{1}{8}x^8 + c$

$f(x) = 2x^6 + \frac{1}{8}x^8 + c$

viii. $f'(x) = (x^4 - 2)(3 - 2x^2)$

$f'(x) = -2x^6 + 3x^4 + 4x^2 - 6$

$f(x) = \frac{-2}{7}x^7 + \frac{3}{5}x^5 + \frac{4}{3}x^3 - 6x + c$

ix. $f'(x) = \frac{15x^{12} - 3x^2 - 8x^5}{x^2}$

$f'(x) = 15x^{10} - 3 - 8x^3$

$f(x) = \frac{15}{11}x^{11} - 3x - \frac{8}{4}x^4 + c$

$f(x) = \frac{15}{11}x^{11} - 3x - 2x^4 + c$

k. Find the equations of the original functions by integrating and finding the value of the constant, c:

i. $\frac{dy}{dx} = 8x^3$ and the original function passes through (1,3)

$y = 2x^4 + c$

Sub. in $(x,y) = (1,3)$

$(3) = 2(1)^4 + c$

Rearrange for c:

$c = 1$

Substitute c back in:

$y = 2x^4 + 1$

ii. $\frac{dy}{dx} = \frac{3}{2}x^2$ and the original function passes through (2,0)

$y = \frac{1}{2}x^3 + c$

Sub. in $(x,y)=(2,0)$

$(0) = \frac{1}{2}(2)^3 + c$

Simplify and rearrange for c:

$c = -4$

Substitute c back in:

$y = \frac{1}{2}x^3 - 4$

iii. $f'(x) = (x + 1)(3x - 1)$ and $f(x)$ passes through $(0,-5)$

$f'(x) = 3x^2 + 2x - 1$

$f(x) = x^3 + x^2 - x + c$

Sub. in $(x,y) = (0, -5)$, where y is equivalent to $f(x)$

$(-5) = (0)^3 + (0)^2 - (0) + c$

Simplify and rearrange for c:

$c = -5$

Substitute c back in:

$f(x) = x^3 + x^2 - x - 5$

iv. $f'(x)= \frac{15x^{12} - 3x^2 - 8x^5}{x^2}$ and $f(x)$ passes through $(0,0)$

$f'(x) = 15x^{10} - 3 - 8x^3$

$f(x) = \frac{15}{11}x^{11} - 3x - 2x^4 + c$

Sub. in $(x,y)=(0,0)$, where y is equivalent to $f(x)$

$(0) = \frac{15}{11}(0)^{11} - 3(0) - 2(0)^4 + c$

Simplify and rearrange for c:

$c = 0$

Substitute c back in:

$f(x) = \frac{15}{11}x^{11} - 3x - 2x^4$

Answers

v. $\frac{dy}{dx} = \frac{3}{4}x^{\frac{1}{2}}$ and the original function passes through $(4,8)$

$y = \frac{1}{2}x^{\frac{3}{2}} + c$

Sub. in $(x,y) = (4,8)$

$(8) = \frac{1}{2}(4)^{\frac{1}{2}} + c$

Simplify and rearrange for c:

$c = 4$

Substitute c back in:

$y = \frac{1}{2}x^{\frac{3}{2}} + 4$

vi. $\frac{dy}{dx} = \sqrt[3]{x}$ and the original function passes through (1,0)

$\frac{dy}{dx} = 8x^{\frac{1}{3}}$

$y = 6x^{\frac{4}{3}} + c$

Sub. in $(x,y) = (1,0)$

$(0) = 6(1)^{\frac{4}{3}} + c$

Simplify and rearrange for c:

$c = -6$

Substitute c back in:

$y = 6x^{\frac{4}{3}} - 6$

vii. $f'(x) = 2x + \sqrt{x} + 3x + 40$ and $f(x)$ passes through $(0,99)$

$f'(x) = 5x + x^{\frac{1}{2}} + 40$

$f(x) = \frac{5}{2}x^2 + \frac{2}{3}x^{\frac{1}{2}} + 40x + c$

Sub. in $(x,y) = (0,-5)$

$(99) = \frac{5}{2}(0)^2 + \frac{2}{3}(0)^{\frac{3}{2}} + 40(0) + c$

Simplify and rearrange for c:

$c = 99$

Substitute c back in:

$f(x) = \frac{5}{2}x2 + \frac{2}{3}x^{\frac{3}{2}} + 40x + 99$

viii. $\frac{dy}{dx} = \frac{x^2 + 2x + 1}{(x+1)}$ and the original function passes through $(10,100)$

$\frac{dy}{dx} = \frac{(x+1)^2}{(x+1)}$

$\frac{dy}{dx} = x + 1$

$y = \frac{1}{2}x^2 + x + c$

Sub. in $(x,y) = (10,100)$

$(100) = \frac{1}{2}(10)^2 + (10) + c$

Simplify and rearrange for c:

$c = 40$

Substitute c back in:

$y = \frac{1}{2}x^2 + x + 40$

ix. $\frac{dy}{dx} = 6x^{-2}$ and the original function passes through $(-12,1)$

$y= -6x^{-1} + c$

Sub. in $(x,y) = (-12,1)$

$(1)= -6(-12)^{-1} + c$

Simplify and rearrange for c:

$c = 12$

Substitute c back in:

$y = -6x^{-1} + 12$

Answers

l. Evaluate the following definite integrals:

i.
$y = \int_{1}^{2} 8x^3\,dx$

$y = [2x^4]^2_1$
$y = [2(2)^4] - [2(1)^4]$
$y = 30$

ii.
$y = \int_{0}^{10} 20x\,dx$

$y = [10x^2]^{10}_0$
$y = [10(10)^2] - [10(0)^2]$
$y = 1000$

iii.
$y = \int_{1}^{3} (2x - 3)(4x^2 + 3)\,dx$

$y=\int_{1}^{3} 8x^3 - 12x^2 + 6x - 9\,dx$
$y=[2x^4 - 4x^3 + 3x^2 - 9x]^3_1$
$y=[2(3)^4 - 4(3)^3 + 3(3)^2 - 9(3)]$
$\quad -[2(1)^4 - 4(1)^3 + 3(1)^2 - 9(1)]$
$y = 62$

iv.
$y = \int_{1}^{2} \sqrt{x} + x^{-2}\,dx$

$y = \int_{1}^{2} x^{\frac{1}{2}} + x^{-2}\,dx$
$y = [\frac{2}{3}x^{\frac{3}{2}} - x^{-1}]^2_1$
$y = [\frac{2}{3}(2)^{\frac{3}{2}} - (2)^{-1}] - [\frac{2}{3}(1)^{\frac{3}{2}} - (1)^{-1}]$
$y = 1.719$

v.
$y = \int_{5}^{6} 8x + 3x^2\,dx$

$y = [4x^2 + x^3]^6_5$
$y = [4(6)^2 + (6)^3] - [4(5)^2 + (5)^3]$
$y = 135$

vi.
$y = \int_{5}^{6} \sqrt{x} - (x^2 - 3)\,dx$

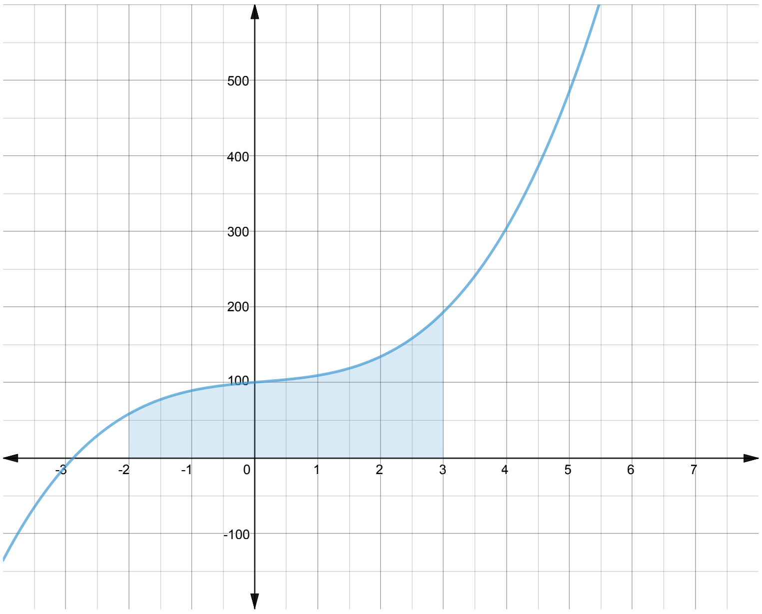
$y = \int_{5}^{6} x^{\frac{1}{2}} - x^2 + 3\,dx$
$y = [\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{3}x^3 + 3x]^{1.2}_0$
$y = [\frac{2}{3}(1.2)^{\frac{3}{2}} - \frac{1}{3}(1.2)^3 + 3(1.2)]$
$\quad -[\frac{2}{3}(0)^{\frac{3}{2}} - \frac{1}{3}(0)^3 + 3(0)]$
$y = 3.900$

m. How do you find the area under a curve?

You evaluate the integral between the two x values which mark the area to be found.

i. The curve $y = 3x^3 - x^2 + 7x + 100$ is shown on the graph below. Find the area shaded, between $x = -2$ and $x = 3$.

Answers



$A = \int_{-2}^{3} 3x^3 - x^2 + 7x + 100 \, dx$

$A = [\frac{3}{4}x^4 - \frac{1}{3}x^3 + \frac{7}{2}x^2 + 100x]^3_{-2}$

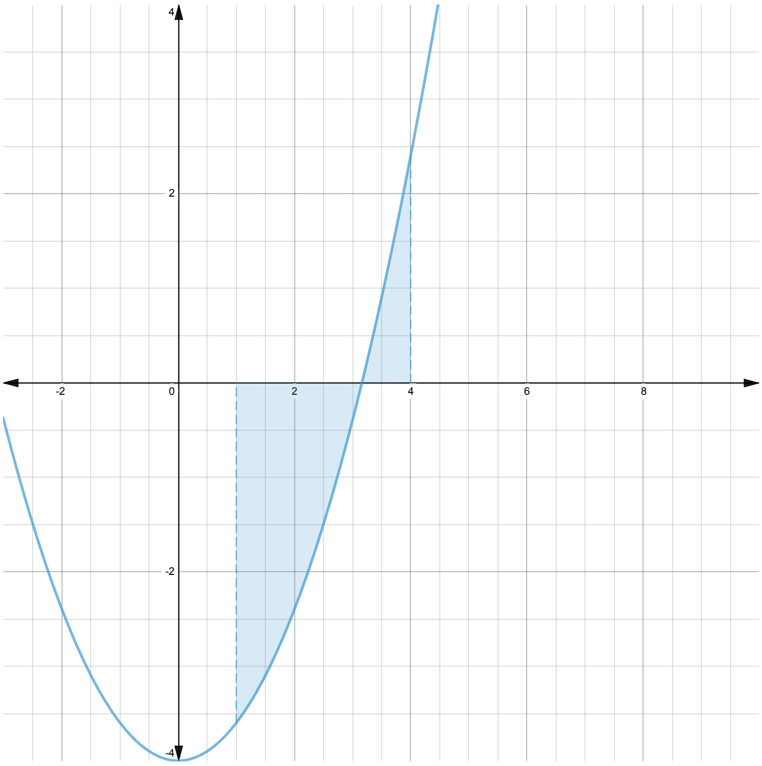
$A = [\frac{3}{4}(3)^4 - \frac{1}{3}(3)^3 + \frac{7}{2}(3)^2 + 100(3)] - [\frac{3}{4}(-2)^4 - \frac{1}{3}(-2)^3 + \frac{7}{2}(-2)^2 + 100(-2)]$

$A = 383.25 - (-171.33) = 554.58$

The shaded area is 554.58.

n. For the following curves determine the shaded area between the curve and the x−axis:

i. $y = 0.4x^2 - 4$ between $x = 1$and $x = 4$. Hint: the root visible is at $x = 3.16$.



We have to calculate the area above the x axis and below the x axis separately and then add them, otherwise they would cancel out.

$$A = \int_{1}^{3.16} 0.4x^2 - 4\ dx + \int_{3.16}^{4} 0.4x^2 - 4\ dx$$

$$A = [\frac{0.4}{3}x^3 - 4x]^{3.16}{}_{1} + [\frac{0.4}{3}x^3 - 4x]^{4}{}_{3.16}$$

$$A = [\frac{0.4}{3}x^3 - 4x]^{3.16}{}_{1} = [\frac{0.4}{3}(3.16)^3 - 4(3.16)] - [\frac{0.4}{3}(1)^3 - 4(1)]$$

$$A = -\ 8.43 - (-3.87)$$

$$A = -\ 4.56$$

This is negative as the area is under the x axis. Since we only care about the size of it, we'll take the value for the area as a positive number – that is, 4.56.

$$[\frac{0.4}{3}x^3 - 4x]^{4}{}_{3.16} = [\frac{0.4}{3}(4)^3 - 4(4)] - [\frac{0.4}{3}(3.16)^3 - 4(3.16)]$$

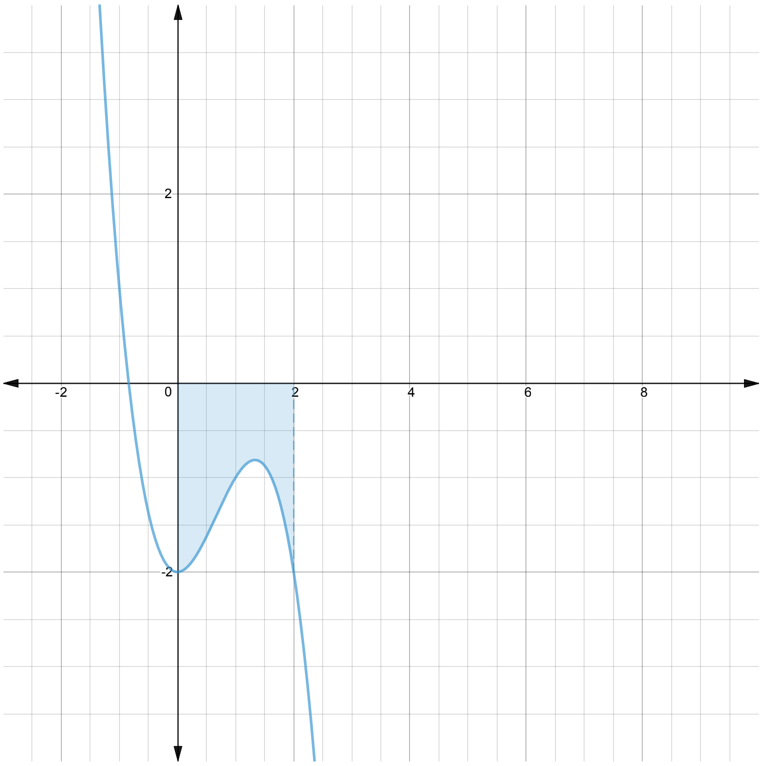
$$A = -7.47 - (-8.43)$$

$$A = 0.96$$

$$A = 4.56 + 0.96$$

$$A = 5.52$$

ii. $y = -\ x^3 + 2x^2 - 2$ between $x = 0$ and $x = 2$



$$A = \int_0^2 -x^3 + 2x^2 - 2\,dx$$

$$A = [-\tfrac{1}{4}x^4 + \tfrac{2}{3}x^3 - 2x]^2_0$$

$$A = [-\tfrac{1}{4}(2)4 + \tfrac{2}{3}(2)^3 - 2(2)] - [-\tfrac{1}{4}(0)^4 + \tfrac{2}{3}(0)3 - 2(0)]$$

$A = -2.67 - 0$

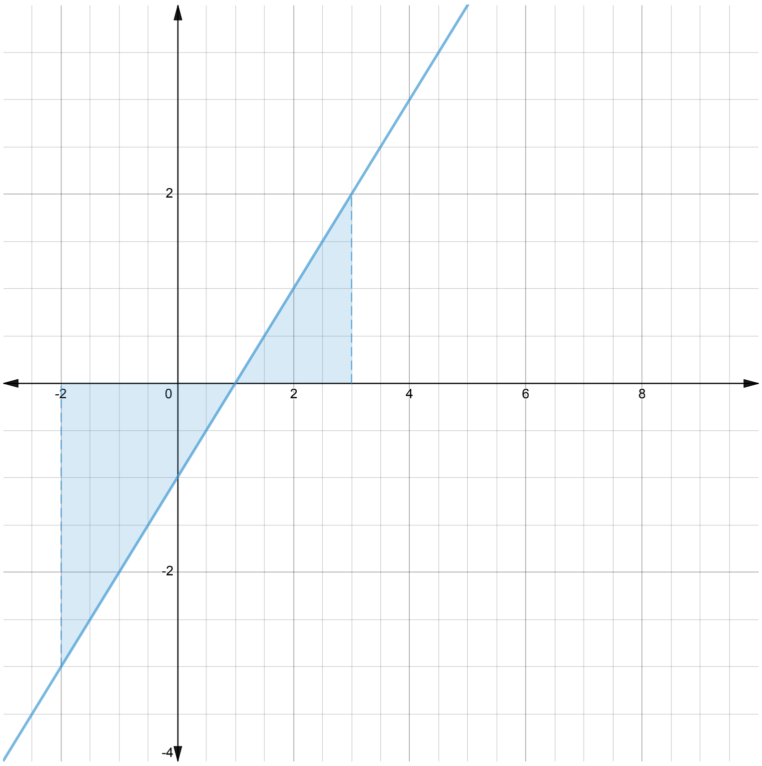
$A = -2.67$

This is negative as the area is below the x axis, but since we're talking about area, we need to switch it to be positive.

$A = 2.67$

iii. $y = x - 1$ between $x = -2$ and $x = 3$

Answers



The line intersects the x axis at $x=1$.

We have to calculate the area above the x axis and below the x axis separately and then add them, otherwise they would cancel out.

$A = \int_{1}^{3} x-1 \, dx + \int_{-2}^{1} x - 1 \, dx$

$A = [0.5x^2 - x]^3{}_1 + [0.5x^2 - x]^1{}_{-2}$

$[0.5x^2 - x]^3{}_1 = [0.5(3)^2 - 3] - [0.5(1)^2 - 1] = 2$

$[0.5x^2 - x]^1{}_{-2} = [0.5(1)^2 - 1] - [0.5(-2)^2 - (-2)] = -0.5 - 4 = -4.5$

This is negative as the area is below the x axis. Since we're talking about area, we need to switch it to positive.

$A = 2 + 4.5 = 6.5$

Section Two
Exam Skills & Mixed Practice

1. Mixed Differentiation and Integration

a. Find the derivative of the function $f(x) = -3x^2 + 4x$.

Because we need to differentiate, we will use the power rule:

$f'(x) = -6x + 4$

b. A function has the derivative function $f'(x) = -6x^2 - 100$. Find the original function.

To get the original function we need to integrate, so we will use the inverse power rule:
$f(x) = -2x3-100x + c$

c. Given that $f(x) = x(2x^2 - 3)$, find the gradient function $f'(x)$.

To get the gradient function we need to differentiate, so we will use the power rule. The equation isn't in a differentiable form, so we first need to expand the brackets:
$f(x) = 2x^3 - 3x$
Differentiating:
$f'(x) = 6x^2 - 3$

d. Given the gradient function is $\frac{dy}{dx} = 4x^3 - 12x^2 + 9$, find the function y.

To get the original function we need to integrate, so we will use the inverse power rule:
$f(x) = x^4 - 4x^3 + 9x + c.$
As we do not have any other information about the original function, we are unable to determine what the value of c is so we leave it as it is.

e. A function f is defined by $f(x) = 2x^2 - ax + 10$, where a is a constant. Find the gradient function $f'(x)$, in terms of x and a.

To get the gradient function we need to differentiate, so we use the power rule. Because a is a constant we can treat 'a' as a number and differentiate as normal:

$f'(x) = 4x - a$

f. Integrate the following function $\frac{dy}{dx} = kx^4 - 2hx$, to find an expression for y, where k and h are constants.

To integrate we will use the inverse power rule. Because 'k' and 'h' are constants we can treat them as a normal number and integrate as normal:
$y = \frac{k}{5}x^5 - hx^2 + c$

2. Tangents and Stationary Points

a. The equation $f(x) = x^2 - 2x + 3$ has one stationary point (turning point). Find the coordinates of this point and state whether it is a maximum or a minimum.

To find turning points, differentiate $f(x)$, set it to zero, and solve for x:
$f'(x) = 2x - 2$
$0 = 2x - 2$
$x = 1$
The turning point is at $x = 1$.
To find the y−coordinate of the turning point, find the value of the original function by substituting $x = 1$ into $f(x)$:
$f(1) = (1)^2 - 2(1) + 3 = 2$
The coordinates of the turning point are (1,2).

Now test whether the turning point is a maximum or a minimum by substituting x−values into f(x), either side of the turning point:
$f(0) = (0)^2 - 2(0) + 3 = 3$
$f(2) = (2)^2 - 2(2) + 3 = 3$
Since these values are greater than the turning point y−value (which was 2), the turning point is a minimum.

b. The function $f(x) = 2x^3 + bx - 4$ has two turning points, at $x = -2$ and $x = 2$. By finding the value of the constant b, find the function $f(x)$:

To find turning points, differentiate and set to zero:
$f'(x) = 6x^2 + b = 0$
We have been told that f(x) has turning points at $x = +/- 2$, so $6x^2 + b = 0$ must hold true for these values. Substitute in $x = 2$ (you could also use $x = -2$ instead) to get the following equation:
$6(2)^2 + b = 0$
Solve for b:
$b = -24$
So the original function is $f(x) = 2x^3 - 24x - 4$.

c. An unknown function $f(x)$ has a gradient function $f'(x) = ax - 18$, where a is an unknown constant. The function $f(x)$ has a turning point at the coordinates $(3, -10)$. By finding the value of the unknown constant a, find the unknown function $f(x)$:

We know the function must have a turning point at $x = 3$, and we can use this information to solve for a. If there is a turning point at $x = 3$, then $f'(3) = 0$. Substituting $x = 3$ below:
$f'(3) = a(3) - 18 = 0$
Solve for a:
$a = 6$
Now we know our gradient function is $f'(x) = 6x - 18$. To get the original function we integrate:
$f(x) = 3x^2 - 18x + c$
Now we have to find c. Since we know f(x) has a turning point at $(3, -10)$, we know it must pass through this point as well. Hence we can find c by substituting in $x = 3$ and $y = -10$, and rearranging:
$(-10) = 3(3)^2 - 18(3) + c$
Solve for c:
$c = 17$
The original function is $f(x) = 3x^2 - 18x + 17$.

d. Find the equation of the tangent to the graph of the function $f(x) = 6x^3 - 5x + 1$ at the point $(-1, 0)$.

Tangent line has the equation type $y = mx + c$. To find m (the gradient), we calculate the gradient of f(x) at the point given. Differentiate:

$f'(x) = 18x^2 - 5$

Substitute $x = -1$ to find gradient at that point:

$f'(-1) = 18(-1)^2 - 5 = 13$

$m = 13$

Now we have $y = 13x + c$, and need to find c. Because the tangent line passes through $(-1, 0)$, we find c by substituting the coordinates into our tangent line equation:

$(0) = 13(-1) + c$

Solve for c:

$c = 13$

So the equation of the tangent to the graph at $(-1,0)$ is $y = 13x + 13$

e. Find the equation of the tangent to the graph of the function $y = 2x^2 - 4x^3 + 2x$ at the point $(2, -20)$.

Tangent line has the equation type $y = mx + c$. To find m (the gradient), we calculate the gradient of $f(x)$ at the point given. Differentiate:

$\frac{dy}{dx} = 4x - 12x^2 + 2$

Substitute $x = 2$ to find the gradient at that point:

$\frac{dy}{dx} = 4(2) - 12(2)^2 + 2 = -38$

$m = -38$

Now we have $y = -38x + c$, and need to find c. Because the tangent line passes through (2, −20), find c by substituting the coordinates into our tangent line equation:

$(-20) = -38(2) + c$

Solve for c:

$c = 56$

So equation of the tangent to the graph at $(2,-20)$ is $y = -38x + 56$

The equation of the tangent to the graph of the function $f(x) = hx^2 + 2$ at the point where $x = 1$ is $y = 4x$. Find the value of the unknown constant, h:

Tangent line has the equation type $y = mx + c$. To find m (the gradient), we calculate the gradient of f(x) at the point given. Differentiate:

$f'(x) = 2hx$

From the tangent equation given, we know the gradient must be 4. Substitute $x = 1$, $f'(x) = 4$and solve for h:

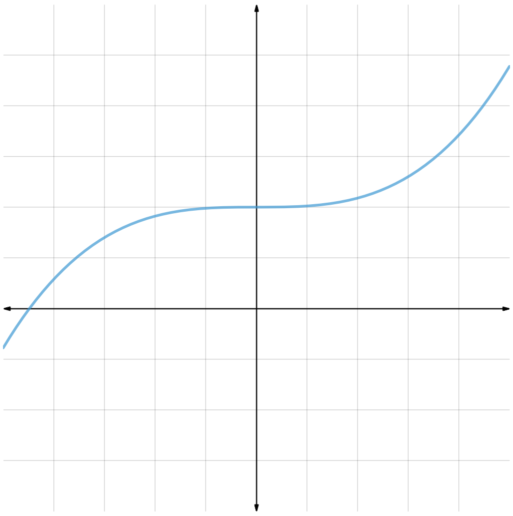
$(4) = 2h(1)$

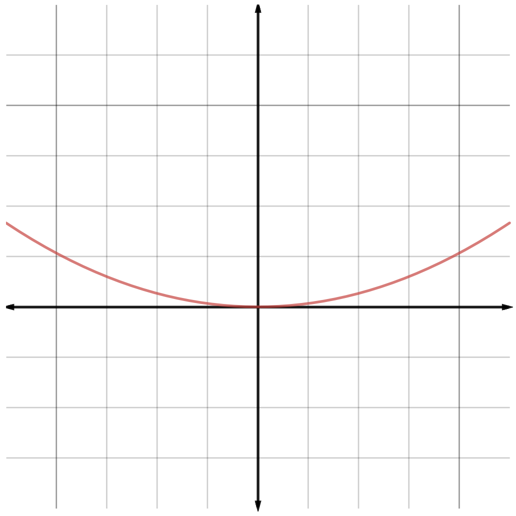
$h = 2$

Answers

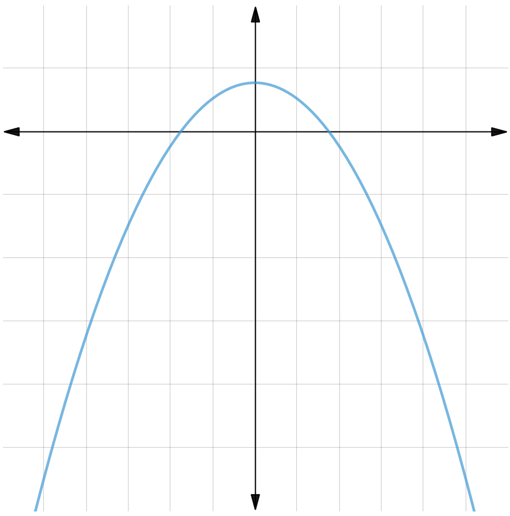
3. Drawing Graphs

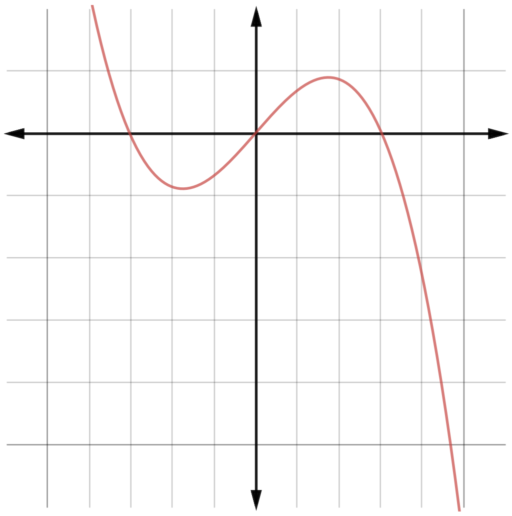
a. The graph $f(x)$ is shown on the diagram below. Sketch $f'(x)$ on the axis below.



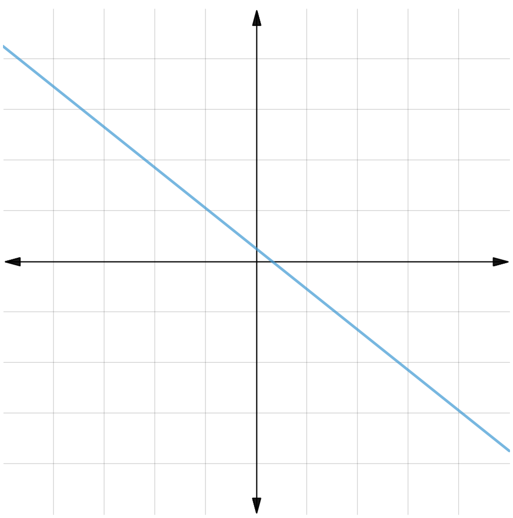


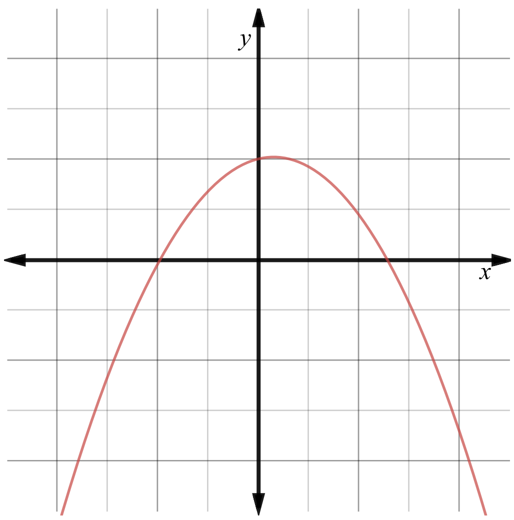
b. $f'(x)$ is shown on the diagram below. Sketch the shape of $f(x)$.





c. $\frac{dy}{dx} = y$ is shown on the graph below. Draw the function y on the axis given.





4. Kinematics

a. The distance a marble runs down a sloped track is given by the equation:
$d(t) = t^2 - 4t$, where t is time in seconds since the marble was released, and d is the distance in meters that the marble has travelled down the track. Show that the acceleration experienced by the marble is constant.

Differentiate distance twice to get acceleration:
$v(t) = 2t - 4$
$a(t) = 2$
So the marble experiences $2ms^{-2}$ of acceleration down the track. Because this value is *independent* of t, it is a constant value. (It always the same value, regardless of t)

b. The velocity v ms−1 of an object t seconds after it passes some fixed point can be described by the function $v(t) = 4t2 - t^3 + 4t + 20$.

i. Find the equation for the acceleration of the object

Differentiate velocity to get acceleration:
$a(t) = 8t - 3t^2 + 4$

ii. Find the acceleration of the object 2 seconds after passing the fixed point.

Using expression for a(t) from the previous question, substitute in t=2:
$a(2) = 8(2) - 3(2)^2 + 4$
$a(2) = 8ms^{-2}$

c. The acceleration (ms^{-2}) of a race car, t seconds after the start of a race, can be modelled by $a(t) = -0.1t^2 + 6$, valid for $0 \leq t \leq 4$. Use calculus to answer the following questions:

i. What should the velocity of the car be at $t = 0$ seconds?

Zero, because the car must be sitting still at the start of a race.

ii. What is the velocity of the car 4 seconds after the race began?

To find velocity, we integrate acceleration:
$v(t) = -\frac{0.1}{3}t^3 + 6t + c$
Knowing that $v = 0$ at $t = 0$, we can substitute these values to find c:
$(0) = -\frac{0.1}{3}(0)^3 + 6(0) + c$
$\Rightarrow \quad c = 0$
So we have an expression for velocity:
$v(t) = -\frac{0.1}{3}t^3 + 6t$
Substitute in $t=4$ seconds, and solve:
$v(4) = -\frac{0.1}{3}(4)3 + 6(4)$
$v(4) = 26.13$
So the velocity of the car after 4 seconds is $26.13\ ms^{-1}$

iii. How far did the car travel in the first 4 seconds of the race?

We can integrate our expression for velocity (that we found in the last question) to get distance:

$d(t) = -\frac{0.1}{12}t^4 + 3t^2 + c$

We know the car won't have travelled anywhere before the race starts, so d=0 when t=0. Substitute these values to find c:

$(0) = -\frac{0.1}{12}(0)^4 + 3(0)^2 + c$

$\Rightarrow \quad c = 0$

So we have an expression for distance:

$d(t) = -\frac{0.1}{12}t^4 + 3t2$

Substitute $t = 4$ seconds, and solve:

$d(4) = -\frac{0.1}{12}(4)^4 + 3(4)^2$

$d(4) = 50.13$

So the car travelled $50.13m$ in the first 4 seconds of the race

d. A football is kicked high up into the air so that t seconds after is kicked, the height above the ground, h meters, is given by $d(t) = 15t - 5t^2$. Use calculus to answer the following questions:

i. At what time did the football reach its maximum height?

To find the maximum height, we will differentiate and set to zero. Talking in terms of kinematics, velocity can be found by differentiating distance:

$v(t) = 15 - 10t$

When the ball is at its maximum height, velocity will be zero:

$(0) = 15 - 10t$

Solve for t:

$t = 1.5$

So the football reached its maximum height 1.5 seconds after being kicked.

ii. What was the maximum height that the football reached?

From the last question, we know the football was at its maximum height at 1.5 seconds, so we can substitute this back into the original d(t) equation to find the maximum height:

$d(1.5) = 15(1.5) - 5(1.5)^2$

$d(1.5) = 11.25$

So the maximum height the football reached was 11.25 meters.

e. Zoe is driving at $25\ ms^{-1}$ and sees a red traffic light 100 meters ahead. She begins braking, causing her car to decelerate at $a(t) = -5\ ms^{-2}$. If she continues breaking at this rate, use calculus to answer the questions below to work out whether Zoe will stop before she reaches the traffic lights.

Answers

i. How many seconds after she begins braking will it take for Zoe's car to come to a stop?

Integrate acceleration to get velocity:
$v(t) = -5t + c$
Define t as time in seconds after Zoe began braking.
Looking back at the original question, Zoe's velocity at $t = 0$ was 25 ms^{-1}.
We can use this to solve for c:
$(25) = -5(0) + c$
$\Rightarrow c = 25$
So we have an expression for velocity:
$v(t) = -5t + 25$
When the car comes to a complete stop, the velocity will be zero. Substitute in v=0 and solve for t:
$(0) = -5t + 25$
$t = 5$
So the car will come to complete stop 5 seconds after Zoe started braking.

ii. What distance will her car have travelled, in the time it took her to come to a stop?

Integrate velocity (from above) to get distance:
$d(t) = -2.5t^2 + 25t + c$
It is easiest to define d(t) as "distance travelled from where Zoe started braking". This means the distance at $t = 0$ is $d = 0$, which can be substituted and used to solve for c:
$(0) = -2.5(0)^2 + 25(0) + c$
$\Rightarrow \quad c = 0$
So we have an expression for distance:
$d(t) = -2.5t^2 + 25t$
To find distance travelled before car came to a stop, substitute $t = 5$ (from the previous question):
$d(5) = -2.5(5)^2 + 25(5)$
$d(5) = 62.5$

iii. Will Zoe's car have stopped on time before she gets to the lights?

The car travels 62.5 meters before coming to a stop. This is less than the distance to the lights (100m), so she will reach a stop before she comes to the traffic lights.

f. Consider Zoe's braking situation from the question above. Assuming that she brakes (decelerates) at a constant rate the whole time, what rate would she need to brake (decelerate) at in order to stop exactly at the traffic lights? (Hint: use an unknown constant k as the acceleration value to represent Zoe's braking, so that $a(t) = -k \ \ ms^{-2}$. Follow similar working as the last question, but keep your answers in terms of the unknown constant k).

Starting with the acceleration equation $a(t) = -k$, integrate to get velocity:

$v(t) = -kt + c$

Remember that she started braking at $25 \ ms^{-2}$, so $v = 25$ when $t = 0$:

$(25) = k(0) + c$

$\Rightarrow c = 25$

So we have an expression for velocity:

$v(t) = -kt + 25$

Set $v = 0$ and solve for t to find how long it takes for the car to come to a rest:

$(0) = -kt + 25$

$t = \frac{25}{k}$

Now integrate $v(t)$ to get $d(t)$:

$d(t) = -\frac{k}{2}t^2 + 25t + c$

Substitute $t = 0, \ d = 0$ and solve for c:

$(0) = -\frac{k}{2}(0)2 + 25(0) + c$

$\Rightarrow \quad c = 0$

So we have an expression for d(t):

$d(t) = -\frac{k}{2}t^2 + 25t$

Finally, substitute in the stopping time $(t = \frac{25}{k}k)$ and the desired stopping distance $(d = 100 \ m)$, and solve for k:

$(100) = -\frac{k}{2}(\frac{25}{k})^2 + 25(\frac{25}{k})$

$100 = -\frac{626}{2k} + \frac{626}{k}$

$100k = -\frac{626}{2} + 625$

$k = 3.125$

So Zoe would have to brake at a constant rate of $3.125 \ ms^{-2}$ if she wanted to stop right at the traffic lights.

5. Rate of Change Mixed Practice

a. A leak is causing a puddle of water to form. The area of the puddle is given by $A = 0.2t^2 + 2$, where t is time in hours since the leak began. Use calculus to find the rate of change of the area of the puddle, 8 hours after the leak began.

$dAdt = 0.4t$

At $t = 8, \frac{dA}{dt} = 0.4(8) = 3.2 \ m^2$ per hour

b. A bouncy castle is inflating at a constant rate of $6m^3$ per minute. After 3 minutes, the volume of the bouncy castle is $20m^3$.

i. Determine an expression for the volume of the bouncy castle.

$\frac{dA}{dt} = 6$
$V = \int 6 \, dt$

$V = 6t + c$
Substituting in $t = 3$ and $V = 20$:
$20 = 6(3) + c$
$c = 2$
$V = 6t + 2$

ii. What was the volume of the bouncy castle at $t = 0$ minutes.

$V = 2m^3$

iii. When fully inflated the bouncy castle takes up a volume of 74m³. How long will it take for the bouncy castle to become fully inflated?

$74 = 6t + 2$
$72 = 6t$
$t = \frac{72}{6} = 12$
It will take 12 minutes to inflate the whole bouncy castle.

c. A triangle is growing in size so that at all times its height is 2 times the length of the base. Find the rate of change of the area of the triangle, with respect to the base, when the area of the triangle is $64cm^2$. (Hint: Area of a triangle $= \frac{1}{2} \times \text{base} \times \text{height}$)

If $h = 2b$ then $A = 0.5 \times b \times 2b$ so $A = b^2$
$\frac{dA}{db} = 2b$
When $A = 64cm^2$
$64 = b^2$ so $b = 8cm$
$\frac{dA}{db} = 2(8) = 16 \frac{cm^2}{cm}$

d. A bubble $(V = \frac{4}{3} \pi r^3)$ is growing. When the volume of the bubble is 20 mm^3, what is the rate of change of the volume of the bubble with respect to the bubble's radius?

First, we need to find the radius when $V = 20\pi$:
$20 = \frac{4}{3} \pi r^3$
$r^3 = 15$ so $r = 2.47mm$
Next we need to find $\frac{dV}{dr}$:
$\frac{dV}{dr} = \frac{4}{3}\pi \times 3r^2 = 4\pi r^2$
When $V = 20, r = 2.47$ so:
$\frac{dV}{dr} = 4(2.47)2 = 76.67 \text{ mm3mm}$

e. What is the rate of change of the volume of a cube with respect to length when it has a volume of 8m^3?

$V = l^3$

$\frac{dV}{dl} = 3l^2$

$8 = l3$

$l = 2$

$\frac{dV}{dl} = 3(2)^2 = 12\text{m}^3/\text{m}$

6. Finding Constants Mixed Practice

a. The equation $f(x) = 4x^2 - 3hx + 2$ has a turning point at $x = 2$. Find the value of h.

$f'(x) = 8x - 3h$

$f'(2) = 8(2) - 3h = 0$

$h = \frac{16}{3}$

b. A curve $f(x) = x^3 - ax + b$ has a turning point at $x = -1$ and passes through the point (2,12). Find the values of constants a and b.

$f'(x) = 3x^2 - a$

$f'(-1) = 3(-1)^2 - a = 0$

So $a = 3$

And $f(x)=x^3 - 3x + b$

$f(2) = (2)^3 - 3(2) + b = 12$

$b = 10$

c. The graph of a function $f(x) = 2x^2 + 5x - z$ has a tangent line $y = 13x - 16$. Find the value of the constant, z.

$f'(x) = 4x + 5$

Gradient of the tangent line $\text{is } 13$:

$4x + 5 = 13$

$\Rightarrow x= 2$

The line $y = 13x - 16$ is tangent to the curve at $x = 2$

The two functions $f(x)$ and y must have the same value at $x = 2$, so we can substitute $x = 2$ into the two equations and make them equal to each other:

$2(2)^2 + 5(2) - z = 13(2) - 16$

$z = 8$

d. A curve can be described by the function $y = kx^3 - x^2 - kx$. If y has a local minimum at $x = 1$, find k.

$\frac{dy}{dx} = 3kx^2 - 2x - k$

$0 = 3k(1) - 2(1) - k$

$k = 1$

Answers

e. Write (in terms of the positive constant, d) the x values for which the function f is increasing, where $f(x) = 3dx^2 - x^3 + 8d^2 x - 1$.

$f'(x) = 6dx - 2x^2 + 8d^2 = 0$

$x^2 - 3dx - 4d^2 = 0$

$(x + d)(x - 4d) = 0 \Rightarrow x = -d$ and $x = 4d$ are the roots

Test values in between: (remembering that d is a positive constant)

$x = -2d \Rightarrow f'(x) = \text{positive}$

$x = 0 \quad\quad \Rightarrow f'(x) = \text{negative}$

$x = 5d \quad \Rightarrow f'(x) = \text{positive}$

$f(x)$ is increasing when $x<-d \text{ or } x>4d$

f. A rectangle is increasing in size such that its base is always k times the size of its height. When the height of the triangle is 4 meters, the area is increasing at a rate of $12 \text{ m}^2/\text{m}$ with respect to the height. What is the value of the constant, k?

$b=kh,$ where $b =$ base of rectangle and $h =$ height of rectangle

Area: $A = b \times h = kh \times h = kh^2$

$\frac{dA}{dh} = 2kh$

$(12) = 2k(4)$

$k = 1.5$

Answers

Section Three
Practice Exam

Question One

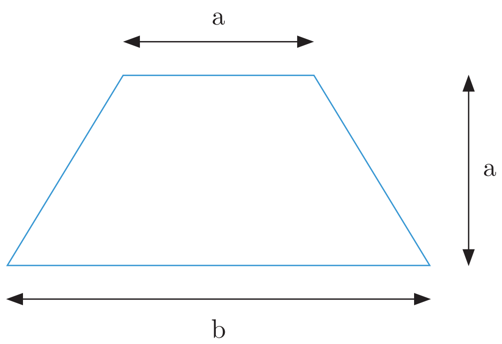
a. A function f is given by $f(x) = 4x(x^2 - 2)$. Find the gradient of the function at the point where $x = 3$.

$f(x) = 4x^3 - 8x$
$f'(x) = 12x^2 - 8$
$f'(3) = 12(3)^2 - 8$
Gradient is 100 when $x = 3$.

b. A function $f(x) = x^3 + kx^2 - 4$ has a maximum at $x = -2$. Find the gradient of f at the point where $x = 1$.

$f'(x) = 3x^2 + 2kx$
We know there's a turning point at x=−2, so we can substitute that in:
$f'(-2) = 3(-2)^2 + 2k(-2) = 0$
$12 - 4k = 0$
$k = 3$
So $f'(x) = 3x^2 + 2(3)x$
$f'(x) = 3x^2 + 6x$
$f'(1)=3(1)2+6(1)$
Gradient is 9 when $x = 1$.

c. A trapezium is growing in size such that length a is always half the size of length b. Find the rate of change of the area of the trapezium, with respect to the length b, when $a = 10$.



General area of a trapezium $= \frac{a+b}{2}h$, but $h = a$ in our case, so $A = \frac{a+b}{2}a$
We also know:
$a = 2b$
$A = \frac{2b+b}{2}(2b)$
$A = 3b^2$
$\frac{dA}{db} = 6b$
When $a = 10$, $b = 20$, as a is always half the size of b
When $b = 20$:
$\frac{dA}{db} = 6(20) = 120$
Rate of change of area is $120\ m^2/m$

d. A function f is defined by $f(x) = x^3 - 2x^2 - x + k$, where k is a positive constant. If f has a tangent line $y = 3x - 4$, find the value of k.

$f'(x) = 3x^2 - 4x - 1$
Find the x values where gradient is same as the tangent line $(\text{which is } m = 3)$:
$3 = 3x^2 - 4x - 1$
$3x^2 - 4x - 4 = 0$
$3x^2 - 6x + 2x - 4 = 0$
$3x(x-2) + 2(x-2) = 0$
$(3x + 2)(x - 2) = 0$
So the tangent line could be tangent to the curve at $x = 2$ or $x = -\frac{2}{3}$
If at $x = -\frac{2}{3}$:
Then $f(x) = 3x - 4$ at $x = -\frac{2}{3}$
$(-\frac{2}{3})^3 - 2(-\frac{2}{3})^2 - (-\frac{2}{3}) + k = 3(-\frac{2}{3}) - 4$
$\Rightarrow k = -\frac{148}{27} = -5.481$ But we were told k is positive, so this can't be right.
Try $x = 2$:
Then $f(x)3x - 4$ at $x = 2$
$(2)^3 - 2(2)^2 - 2 + k = 3(2) - 4$
$k = 4$

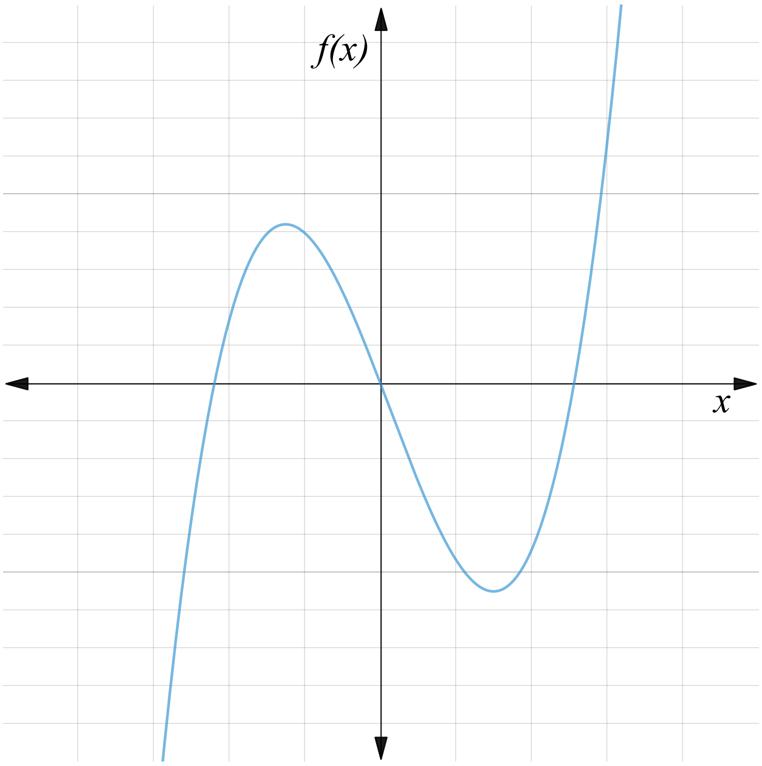
e. Zert is engineering a unominium fuel cell for her intergalactic space yacht. The fuel cell has a cylindrical shape, and to keep the cell stable she has calculated that the sum of the height and the circumference of the circular cross section must be 10cm. To maximise the amount of unominium she can fit into one cell, use calculus to find the maximum volume of the fuel cell.

Volume of a cylinder is:
$V = \pi r^2 h$
To make the sum of the height and the circumference of the circular cross–section less than 10 cm, the restraint is: $h + 2\pi r \leq 10$,
But for max volume set $h + 2\pi r = 10$
$\Rightarrow h = 10 - 2\pi r$
Sub h into the volume formula to get:
$V = r\pi^2(10 - 2\pi r)$
$V = 10\pi r^2 - 2\pi^2 r^3$
Differentiate with respect to r and set to zero to maximise:
$\frac{dV}{dr} = 20\pi r - 6\pi^2 r^2 = 0$
$r(20\pi - 6\pi^2 r) = 0$
This gives two possible solutions – one when $r = 0$ and one when $20\pi - 6\pi^2 r = 0$
$r = 0$ is not a valid answer (as you can't have a radius of zero) so we consider the only other solution:
$20\pi - 6\pi^2 r = 0$
$r = \frac{20}{3\pi}$
So maximum volume is when $r = \frac{20}{3\pi}$
Maximum volume is:
$V = 20\pi(\frac{20}{3\pi})^2 - 2\pi^2(\frac{20}{3\pi})^3$
$V = 411.8 \text{ cm}^3$

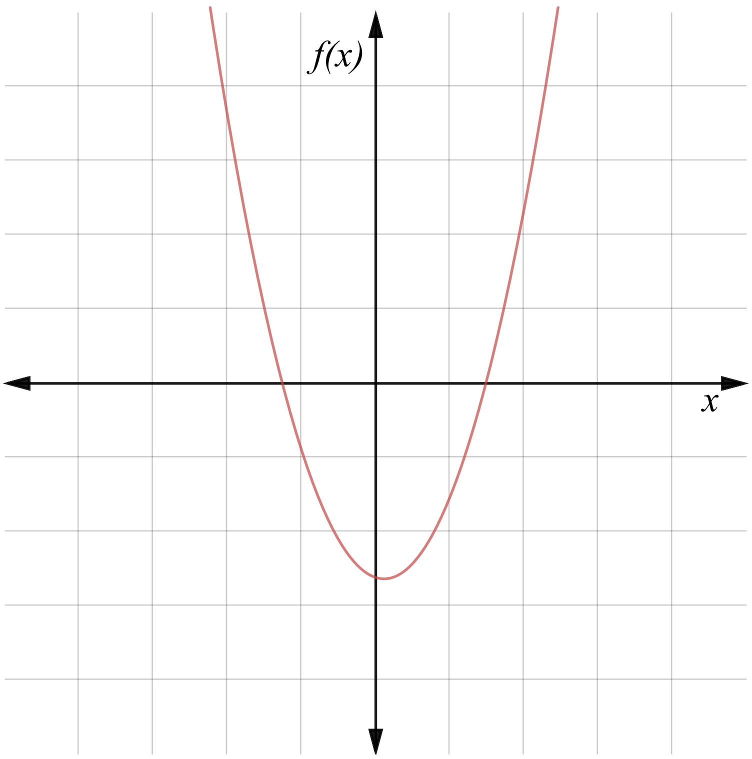
Answers

Question Two

a. The diagram below shows the graph of the function $y = f(x)$.



Sketch the graph of the gradient function $y = f'(x)$ on the axes below.
Both sets of axes have the same scale.



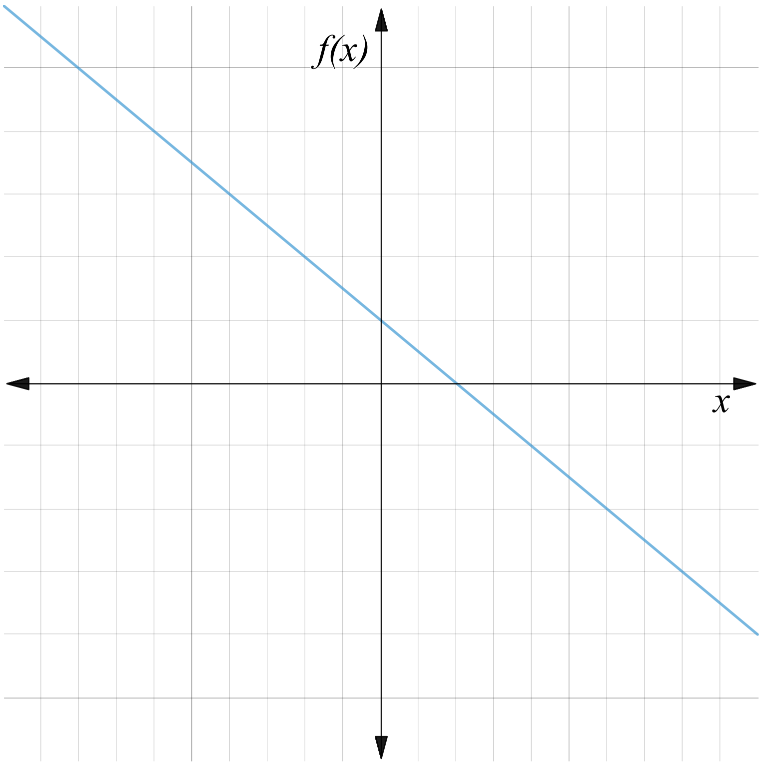
b. The function $f(x) = 2x^3 + kx^2 - 6x + 4$ has a turning point at $x = 1$. Find the value of k.

$f'(x) = 6x^2 + 2kx - 6$
Turning point at x=1 means:
$f'(1) = 6(1)^2 + 2k(1) - 6 = 0$
$k = 0$

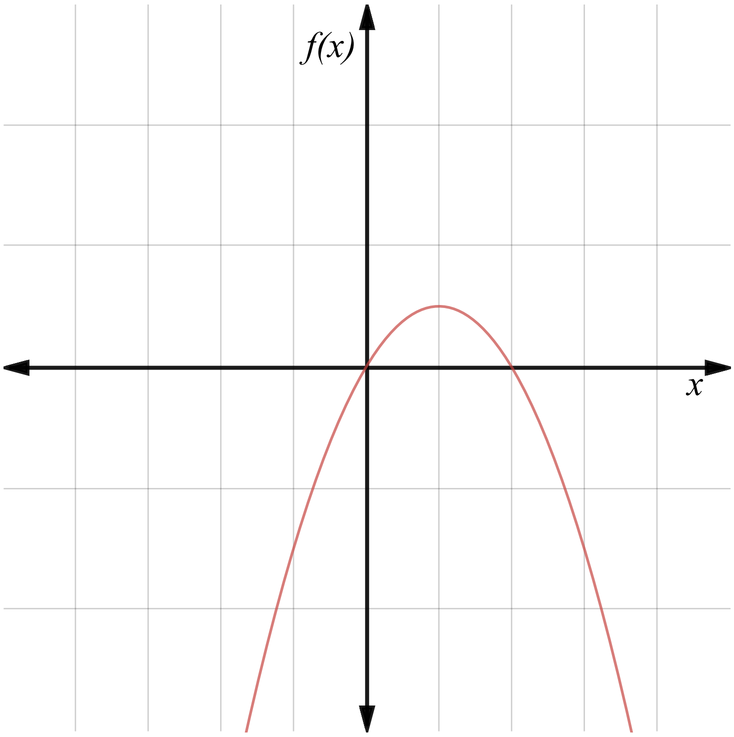
c. The gradient function of some curve is $\frac{dy}{dx} = 6x^2 - 4x$. If the curve passes through the point (2,6), find the equation for the curve.

Integrate:
$y = 2x^3 - 2x^2 + c$
Solve for c:
$(6) = 2(2)^3 - 2(2)^2 + c$
$c = -2$
$y = 2x^3 - 2x^2 - 2$

d. The diagram below show the graph of the function $y = f'(x)$.



Sketch the graph of the original function $y = f(x)$ on the axes below, given that $f(x)$ passes through the point (0,0).
Both sets of axes have the same scale.



e. The acceleration of Zert's space–yacht t seconds after passing the first checkpoint can be given by the function $a(t) = 10 - 3t$. Zert passes the second checkpoint, which is 200m after the first checkpoint, 2 seconds later (at time $t = 2$). What speed was Zert travelling at when she passed the first checkpoint?

Integrate to get velocity:

$v(t) = 10t - \frac{3}{2}t^2 + C$

Integrate again to get distance:

$d(t) = 5t^2 - \frac{1}{2}t^3 + Ct + D$

At time $t = 0$, Zert had travelled no distance $(d = 0)$:

$(0)=5(0)^{\prime 2} - \frac{1}{2}(0)^3 + C(0) + D$

$\Rightarrow$ D=0

So we have: $d(t) = 5t^2 - \frac{3}{2}t^3 + Ct$

At time $t = 2$, Zert had travelled 200m $(d = 200)$:

$(200) = 5(2)^2 - \frac{1}{2}(2)^3 + C(2)$

$C = 92$

So now we can go back to the velocity equation : $v(t) = 10t - \frac{3}{2}t^2 + 92$

At $t = 0$:

$v(t) = 10(0) - \frac{3}{2}(0)^2 + 92$

$v(t) = 92 \text{ m/s}$

Zert was travelling 92 m/s when she passed the first checkpoint.

Question Three

a. A function f is described by $f(x) = 6 - 5x^4 + 2x^3 - 4x^2$.
Find the gradient of the function f at the point where $x = 2$

$f'(x) = -20x^3 + 6x^2 - 8x$
$f'(2) = -20(2)^3 + 6(2)^2 - 8(2)$
$f'(2) = -152$
Gradient is 152 at the point where $x = 2$

b. Show that the line $y = 10x - 1$ is tangent to the graph $f(x) = 2x^3 + 4x - 5$.

$f'(x) = 6x^2 + 4$
Gradient of tangent line is 10, so find points where $f'(x) = 10$:
$10 = 6x^2 + 4$
$x = \pm\sqrt{1} \Rightarrow x = 1 \text{ and } x = -1$
We don't know which x–value the line could be tangent to the curve, so must try both:
Try x = 1:
At $x = 1$, $f(x)$ must equal y
$2(1)^3 + 4(1) - 5 = 10(1) - 1$
$1 = 9$
Because they aren't equal, the tangent isn't at $x = 1$
Try x = – 1
At $x = -1$, $f(x)$ must equal y
$2(-1)^3 + 4(-1) - 5 = 10(-1) - 1$
$-11 = -11$
So therefore the line is tangent, at $x = 1$

c. Zert has found that cooking her fuel cells for different amounts of time changes the amount of energy they can hold. The charge Q [MegaJoules] from cooking a fuel cell for T minutes is given by $Q(T) = 150 + 2T(50 - T)$.

If Zert wants her cells to hold the maximum amount of energy, how long should she cook them for?

To maximise, differentiate and set to zero:
$Q = 150 + 100T - 2T^2$
$\frac{dQ}{dT} = 100 - 4T = 0$
$4T = 100$
$T = 25$ minutes

Answers

d. For the curve given by $f(x) = x^2(2x - x^2)$, find the coordinates of the local maximum.

$f(x) = 2x^3 - x^4$
$f'(x) = 6x^2 - 4x^3$
Set to zero to find turning points:
$6x^2 - 4x^3 = 0$
$2x^2(3 - 2x) = 0$
So turning points are at $x = 0$ and $x = 1.5$
Test which are maximums by testing points around the answers. Try x= −1, 1, 2
$f'(-1) = 6(-1)^2 - 3(-1)^3 = 9 \Rightarrow$ positive
$f'(1) = 6(1)^2 - 3(1)^3 = 3 \Rightarrow$ positive
$f'(2) = 6(2)^2 - 3(2)^3 = -8 \Rightarrow$ negative

Because the gradient changed from positive to negative between our test points of $x = 1$ and $x = 2$, we know that the turning point at $x = 1.5$ (which is in between those two test points) must be a maximum.
Height at $x = 1.5$ is:
$f(1.5) = 2(1.5)^3 - (1.5)^4$
$f(1.5) = 1.69$
Coordinates of the maximum are (1.5,1.69)

e. For what value(s) of k is the function $f(x) = 3x^2 - kx + 7$ decreasing when $x < 1$?

$f'(x) = 6x - k$
$6x - k < 0$
So the function is decreasing when $x < \frac{k}{6}$
To ensure function is decreasing when $x<1$, we need $\frac{k}{6} \geq 1$
For values $k \geq 6$, $f(x)$ will be decreasing when $x<1$
(Give merit for $k = 6$ as an answer)

Answers