Complex Numbers
ANSWERS

Surds

Surds

STOP AND CHECK (PAGE 4)

- Because there's always the positive and the negative square root. When we multiply a negative number by itself, the negative goes away and it becomes positive.
- The word "surd" refers to a square root which doesn't give us a nice number. We use surds instead of decimal places because it makes our answers neater and more accurate.

Simplifying Surds

STOP AND CHECK (PAGE 5)

- $$\begin{aligned}\sqrt{50} &= \sqrt{25 \times 2} \\ &= \sqrt{25} \times \sqrt{2} \\ &= 5\sqrt{2}\end{aligned}$$
  Note that we picked 2×25 instead of 5×10 or some other choice of two numbers because 25 is a perfect square, and so we can get a nice simplification.

Compound Surds and Adding them Together

STOP AND CHECK (PAGE 6)

- A surd is a single term with a square root on it. A compound surd consists of multiple surd and non-surd terms.
- $(3 + 2\sqrt{3}) + (1 + \sqrt{3}) = 3 + 2\sqrt{3} + 1\sqrt{3}$
  $= (3 + 1) + (2 + 1)\sqrt{3}$
  $= 4 + 3\sqrt{3}$

Multiplying Compound Surds

STOP AND CHECK (PAGE 6)

- To multiply out surds, we first expand out the brackets using FOIL. We then collect any like terms, and then we simplify if possible.

Difference of Two Squares and Conjugates

STOP AND CHECK (PAGE 7)

- The conjugate of a compound surd is the exact same as the original, but with the sign in front of the surd switched. So $5 + \sqrt{19}$ has the conjugate $5 - \sqrt{19}$ and vice versa.
- $(a + b\sqrt{c})(a - b\sqrt{c}) = a^2 + b^2c$

Dividing Compound Surds

STOP AND CHECK (PAGE 8)

- To divide by a compound surd, we first find the conjugate, then multiply the top and bottom of our fraction by the conjugate.

Surds

QUICK QUESTIONS (PAGE 8)

- $(3 + \sqrt{5})(2 - \sqrt{5}) = 6 - 3\sqrt{5} + 2\sqrt{5} - 5$
  $= 1 - 1\sqrt{5}$

- $(5 - \sqrt{2})(5 + \sqrt{2}) = 25 + 5\sqrt{2} - 5\sqrt{2} - 2$
  $= 23$

- $\frac{(3 + 3\sqrt{2})}{(1 + \sqrt{5})} = \frac{(3 + 3\sqrt{2})}{(1 + \sqrt{5})} \times \frac{(1 - \sqrt{5})}{(1 - \sqrt{5})}$
  $= \frac{3 - 3\sqrt{5} + 3\sqrt{2} - 3\sqrt{2 \times 5}}{1 - \sqrt{5} + \sqrt{5} - 5}$
  $= \frac{3 - 3\sqrt{5} + 3\sqrt{2} - 3\sqrt{10}}{-4}$
  $= \frac{3}{4}(1 - \sqrt{2} + \sqrt{5} - \sqrt{10})$

Quadratic Equations

Solving Quadratic Equations

STOP AND CHECK (PAGE 8)

- The quadratic formula is:

$$x = \frac{-b \pm \sqrt{b^{2} - 4ac}}{2a}$$

- The discriminant is:

$$b^{2} - 4ac$$

- A quadratic equation is an equation which we're trying to solve, while the quadratic formula is a way to calculate the possible solutions of x for that equation. We can find the quadratic equation on our formula sheet.

Solving When the Discriminant is Negative

STOP AND CHECK (PAGE 10)

- Quadratics with a positive or 0 discriminant have real roots. These can be solved as we did for Level 2. A quadratic with a negative discriminant needs the square root of a negative number. This is done using the definition:

$$\sqrt{-1} = i$$

Factorising the Un-Factorisable

STOP AND CHECK (PAGE 11)

- Find the roots, r_1 and r_2, using the quadratic equation and the fact that $\sqrt{-1} = i$
  Then write the quadratic in the form:

$$(x - r_1)(x - r_2)$$

Quadratic Equations

QUICK QUESTIONS (PAGE 11)

- $x = \frac{-6 \pm \sqrt{6^2 - 4 \times 1 \times 10}}{2 \times 1}$
  $= \frac{-6 \pm \sqrt{-4}}{2}$
  $= \frac{-6 \pm 2i}{2}$
  $= -3 \pm i$
  So then:
  $r_1 = -3 + i$
  $r_2 = -3 - i$
- $x^2 + 6x + 10 = (x - (-3 + i)(x - (-3 - i)$
  $= (x + 3 - i)(x + 3 + i)$

Using Complex Numbers

Addition and Multiplication of Complex Numbers

STOP AND CHECK (PAGE 12)

- Assuming the rectangular form, we add numbers without i and with i separately, collecting like terms.

Complex Conjugates and Dividing Complex Numbers

STOP AND CHECK (PAGE 13)

- Multiply both the numerator and denominator by the complex conjugate of the denominator.

Actual Algebra of Complex Numbers

STOP AND CHECK (PAGE 14)

- The first step is to replace z with a + b*i* in the equation:

$$a + bi + 3 + 2i = 5i$$

Next, equate the real terms and the imaginary terms:

Real: $a + 3 = 0$
Imaginary: $b + 2 = 5$

Solve these equations:

$$a = -3$$
$$b = 3$$

Now write the final answer for z:

$$z = a + bi$$
$$z = -3 + 3i$$

Using Polar Forms

Complex Numbers in Polar Form

STOP AND CHECK (PAGE 16)

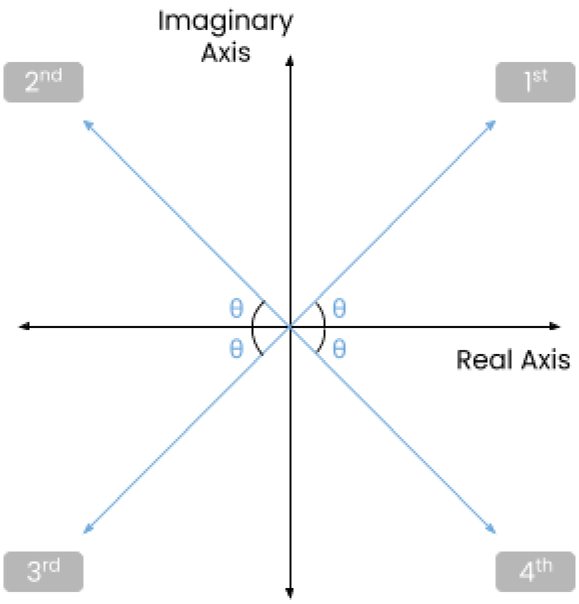
- To draw a number z = a + *i*b, plot the coordinates (a,b) on the Argand diagram and draw an arrow from the origin (0,0) to this point.
- |z| is the length of the arrow (we can also use r), θ is the angle of the arrow from the real axis, $cis(\theta)$ means $cos(\theta) + isin(\theta)$.

Converting from Rectangular to Polar

STOP AND CHECK (PAGE 18)

- $\theta = tan^{-1}\frac{b}{a}$

Note: The quadrants are labelled as follows:



- First quadrant (top right): $arg(z) = \theta$
- Second quadrant (top left): $arg(z) = \pi - \theta$
- Third quadrant (bottom left): $arg(z) = \pi + \theta$
- Fourth quadrant (bottom right): $arg(z) = -\theta$

Multiplying Complex Numbers and De Moivre's Theorem

STOP AND CHECK (PAGE 18)

- $(|z|cis(\theta))^n = |z|^n cis(n\theta)$

Dividing Complex Numbers & Finding Roots of Complex Numbers

STOP AND CHECK (PAGE 21)

- The length of the root is $\sqrt{|z|}$
- The two angles are $\frac{\theta}{2}$ and $\frac{\theta}{2} + \pi$

nth Roots

STOP AND CHECK (PAGE 23)

- A locus is a curve in the real x/y plane.
- z is a variable, so there are usually multiple complex numbers that satisfy the locus restriction.
- The range of the variable z is restricted by the locus equation - only certain types of complex numbers will work.

Using Polar Form

QUICK QUESTIONS (PAGE 23)

- $|z| = \sqrt{1^2 + 2^2}$
  $|z| = \sqrt{1 + 4}$
  $|z| = \sqrt{5}$
  $\theta = tan^{-1}\frac{2}{1}$
  $\theta = 63.4349488$ rounded to 63°
  As $1 + 2i$ is in the first quadrant, $arg(z) = \theta$which is 63°
  $z = \sqrt{5}cis(63°)$

- $|z| = \sqrt{1^2 + 5^2}$
  $|z| = \sqrt{1 + 25}$
  $|z| = \sqrt{26}$
  $\theta = tan^{-1}\frac{5}{1}$
  In degrees:
  $\theta = 78.6900675$ rounded to 79°
  As $-1 + 5i$ is in the second quadrant, $arg(z) = 180 - \theta$
  $180 - 79 = 101°$
  $z = \sqrt{26}cis(101°)$
  In radians:
  $\theta = 1.3734$
  As $-1 + 5$ is in the second quadrant $arg(z) = \pi - \theta$
  $\pi - 1.37 = 1.77rad$

$z = \sqrt{26}cis(1.77)$

- We converted $-1 + 5i$ to polar form in our last question, so we'll use that again here.

  $(-1 + 5i)^{7} = (\sqrt{26}cis(1.77))^{7}$

  $= (\sqrt{26})^{7}cis(1.77 \times 7)$

  $= (\sqrt{26})^{7}cis(12.4)$

  $= 89620cis(12.4)$

  $= 89620(cos(12.4)) + isin(12.4))$

  $= 88382 - 14821i$

- $1 = 1cis(0)$

  $w_1 = \sqrt[3]{1}cis\left(\frac{0}{3}\right)$

  $w_1 = 1cis(0)$

  $w_1 = 1$

  $w_2 = \sqrt[3]{1}cis\left(\frac{0}{3} + 1 \times \frac{2\pi}{3}\right)$

  $w_2 = 1cis\left(\frac{2\pi}{3}\right)$

  $w_2 = cis\left(\frac{2\pi}{3}\right)$

  $w_3 = \sqrt[3]{1}cis\left(\frac{0}{3} + 2 \times \frac{2\pi}{3}\right)$

  $w_3 = 1cis\left(\frac{4\pi}{3}\right)$

  $w_3 = cis\left(\frac{4\pi}{3}\right)$

- $1 + i = \sqrt{2}cis\left(\frac{\pi}{4}\right)$

  $w_1 = \sqrt[4]{\sqrt{2}}cis\left(\frac{\pi}{4}\right)$

  $w_1 = \sqrt[8]{2}cis\left(\frac{\pi}{4}\right)$

  $w_2 = \sqrt[4]{\sqrt{2}}cis\left(\frac{\pi}{4} + 1 \times \frac{2\pi}{4}\right)$

  $w_2 = \sqrt[8]{2}cis\left(\frac{3\pi}{4}\right)$

  $w_3 = \sqrt[4]{\sqrt{2}}cis\left(\frac{\pi}{4} + 3 \times \frac{2\pi}{4}\right)$

  $w_3 = \sqrt[8]{2}cis\left(\frac{5\pi}{4}\right)$

  $w_4 = \sqrt[4]{\sqrt{2}}cis\left(\frac{\pi}{4} + 3 \times \frac{2\pi}{4}\right)$

  $w_4 = \sqrt[8]{2}cis\left(\frac{7\pi}{4}\right)$

Polynomials

Remainders and Factors

STOP AND CHECK (PAGE 25)

- The Remainder Theorem: The remainder R when dividing a polynomial f(x) by (x-a) is R = f(a).
- The Factor Theorem: If f(x) is a polynomial and f(a) = 0 (that is, the remainder according to the remainder theorem is 0) then (x-a) is a factor of f(x).

Fundamental Theorem of Algebra and Conjugate Root Theorem

STOP AND CHECK (PAGE 26)

- The Fundamental Theorem of Algebra tells us that all polynomials of degree n must have exactly n roots. For example, a quintic (a polynomial with x^5 as its largest power) is degree 5 so must have exactly 5 roots.
- The conjugate root theorem tells us that if we have a real polynomial (with no i's in the polynomial itself), and that one root is a complex root, then one of the other roots must be the complex conjugate.

Factoring Cubics Given Complex Roots

STOP AND CHECK (PAGE 28)

- Let's look at $x = 3$ first to make sure we've got the process for non-complex roots.
  Applying the factor theorem, we know that if we substitute $x = 3$ into our polynomial, it will be equal to 0. Then we can solve for A as follows:
  $x^3 - Ax^2 + 17x - 15 = 0$
  $3^3 - A(3)^2 + 17(3) - 15 = 0$
  $27 - 9A + 51 - 15 = 0$
  $9A = 63$

$A = 7$

Which matches what we said before!

Let's take a look at the root $x = 2 - i$. We'll follow the exact same process; it'll just be a bit more complicated since we'll have complex numbers to deal with.

$x^3 - Ax^2 + 17x - 15 = 0$

$(2 - i)^3 - A(2 - i)^2 + 17(2 - i) - 15 = 0$

Expand that out to get:

$(2 - 11i) - (3A - 4Ai) + (34 - 17i) - 15 = 0$

Then pull together like terms (grouping by reals and imaginaries) to get:

$(21 - 3A) + (- 28 + 4A)i = 0$

Now we know that on the right hand side, we have 0 reals and 0 imaginary numbers. So we can pick either of our brackets to set equal and solve for A:

Reals: $21 - 3A = 0$

$3A = 21$

$A = 7$

Imaginary: $- 28 + 4A = 0$

$4A = 28$

$A = 7$

Which both match!

Factorising Cubics Given Real Roots

STOP AND CHECK (PAGE 29)

- So we have the equation

  $(4 - p)p = 5$

  We can notice that it is a quadratic, and so we'll expand the left hand side and rearrange so that the right hand side is 0.

  $4p - p^2 = 5$

  $- p^2 + 5p - 5 = 0$

  Now we can use the quadratic formula or a graphics calculator to get the answers $x = 2 - 1,\ x = 2 + 1$

Polynomials

QUICK QUESTIONS (PAGE 29)

- $x^2 + 5x + k = (x - r_1)(x - r_2)$

  $= (x - r_1)(x - 6r_1)$ since one root is 6 times the other.

  Expand out to get:

  $x^2 + 5x + k = x^2 - 7r_1x + 6r_1^2$

  Now the matching coefficients must be equal, so:

  $5 = -7r_1$

  $r_1 = -\frac{5}{7}$

  Again, since coefficients must be equal, substitute our value of r_1 to get:

  $k = 6\left(-\frac{5}{7}\right)^2$

  $k = 6 \times \frac{25}{49}$

  $k = 3.06122$ rounded to 3.06 (2 d.p.)

- $(-2)^3 + 2(-2)^2 - (-2) - 2 = -8 + 8 + 2 - 2$

  $= 0$

  So, by the Factor Theorem, $(x + 2)$ is a factor of $x^3 + 2x^2 - x - 2$

- $(-3)^3 + 2(-3)^2 = -27 + 18 + 3 - 2$

  $= -8$

  So, by the Factor Theorem $(x + 3)$ is not a factor of $x^3 + 2x^2 - x - 2$and by the Remainder Theorem, the remainder when dividing $x^3 + 2x^2 - x - 2$ by $(x + 3)$ is -8.

Solving Complex Equations

Single Valued Complex Equations

STOP AND CHECK (PAGE 32)

- Firstly, we want to multiply by the complex conjugate. We'll focus on the left hand side first.

$$\frac{4+6i}{10+2i} \times \frac{10-2i}{10-2i} = \frac{40-8i+60i+12}{100+4}$$
$$= \frac{52+52i}{104}$$
$$= \frac{1}{2} + \frac{1}{2}i$$

Now we'll set the left hand side to the right hand side, and notice that the real components must be equal and the imaginary components must be equal. We can then use that information to solve for k.

$$\frac{1}{2} + \frac{1}{2}i = k + ki$$
$$k = \frac{1}{2}$$

- Remember that $|a + bi| = \sqrt{a^2 + b^2}$. Applying this knowledge we get:

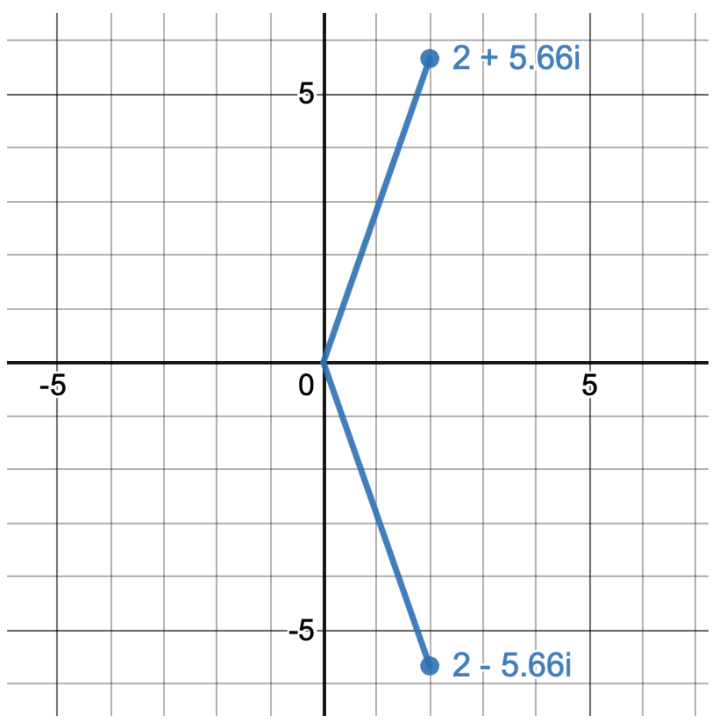
$$|2 + ki| = \sqrt{2^2 + k^2} = 6$$
$$\sqrt{4 + k^2} = 6$$

We'll square both sides so that it's easier to work with:

$$4 + k^2 = 36$$
$$k^2 = 32$$
$$k = \pm\sqrt{32} \approx 5.66\ (2dp)$$

It's important to remember the $\pm$ sign.

This is what our two complex numbers look like on an Argand diagram:



Complex Proofs

STOP AND CHECK (PAGE 33)

- To show it's purely real, we need to show that its real component is 0. To show it's purely imaginary, we need to show that its imaginary component is 0.
- It's a good idea to start by letting $z = a + bi$ and $w = c + di$. This way we can get a bit more information about the components. Then:

  $\sqrt{a^2 + b^2} = \sqrt{c^2 + d^2} = 1$

  We can square everything to get:

  $a^2 + b^2 = c^2 + d^2 = 1$

  Which gives us a lot of information about the components of the two numbers (mainly that a, b, c, and d are all individually less than 1).

Solving Loci Problems

STOP AND CHECK (PAGE 33)

- Firstly, substitute in z=x+yi:

  $|x + yi + i| = 3$

  Then we apply the modulus:

  $\sqrt{x^2 + (y + 1)^2} = 3$

  Now we square both sides and work out the sort of loci:

  $x^2 + (y + 1)^2 = 9$

  This is a circle, which we know since circles take the form $(x - a)^2 + (x - b)^2 = r^2$ which is a circle of radius r centred on the point (a, b).

- Substitute in z=x+yi:

  $|x + yi| = |x + yi + 1|$

  Apply the modulus:

  $\sqrt{x^2 + y^2} = \sqrt{(x + 1)^2 + y^2}$

  Square both sides and simplify:

  $x^2 + y^2 = (x + 1)^2 + y^2$

  $x^2 = (x + 1)^2$

  $= x^2 + 2x + 1$

  $2x + 1 = 0$

$x = -\frac{1}{2}$

Therefore this loci describes a vertical line at $x = -\frac{1}{2}$.

Solving Complex Equations

QUICK QUESTIONS (PAGE 32)

- Our first step is to multiply by the complex conjugate to clean things up.

  $\frac{m+4i}{1+mi} \times \frac{1-mi}{1-mi} = \frac{m-m^2i+4i+4m}{1+m^2}$

  $= \frac{5m}{1+m^2} + \frac{-m^2+4}{1+m^2}i$

  Now since we know z is a real number, the imaginary part must be 0. Therefore we can set the imaginary part equal to 0 to solve for m.

  $\frac{-m^2+4}{1+m^2} = 0$

  $-m^2 + 4 = 0$

  $m^2 = 4$

  $m = \pm 2$

- The first assumption we can see is that z_1 and z_2 have the same length. This can be written:

  $|z_1| = |z_2|$

  Or alternatively,

  $\sqrt{a^2 + b^2} = \sqrt{c^2 + d^2}$ where $z_1 = a + bi$ and $z_2 = c + di$

  Now let's write the second equation with rectangular form:

  $\frac{z_1+z_2}{z_1-z_2} = \frac{a+bi+c+di}{a+bi-c-di}$

  $= \frac{(a+c)+(b+d)i}{(a-c)+(b-d)i}$ (grouping reals and imaginary together)

  This isn't very nice to deal with yet since we don't like imaginary numbers on the bottom, so we'll multiply by the complex conjugate and simplify:

  $\frac{(a+c)+(b+d)i}{(a-c)+(b-d)i} \times \frac{(a-c)-(b-d)i}{(a-c)-(b-d)i} = \frac{(a^2+b^2-c^2-d^2)+(2ad-2bc)i}{(a-c)^2+(b-d)^2}$

  The question says to show that it is purely imaginary. This means we're looking to try and show the real part is 0. A good place to start here is by looking at the real part and seeing what we can substitute in:

  $a^2 + b^2 - (c^2 + d^2)$

Looking at this, we can see it's very similar to the first equation we came up with. If we know $\sqrt{a^2 + b^2} = \sqrt{c^2 + d^2}$, then squaring both sides we get that $a^2 + b^2 = c^2 + d^2$. So we can substitute in to our equation to get:

$a^2 + b^2 - (a^2 + b^2) = 0$

Which is what we wanted! We've proved that the real component is 0 and so it must be imaginary.

- Substitute in $z = x + yi$ and group together real/imaginary to get:

  $|x - 2 + (y - 1)i| = 4$

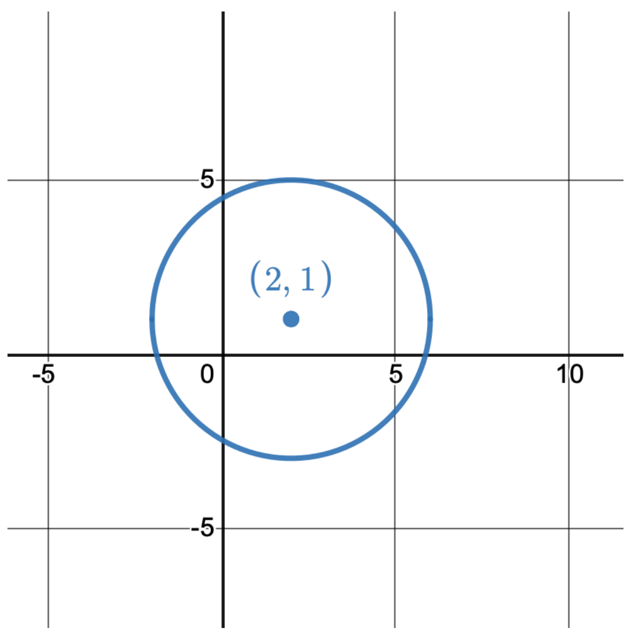
  Use definition of modulus:

  $\sqrt{(x - 2)^2 + (y - 1)^2} = 4$

  Square both sides:

  $(x - 2)^2 + (y - 1)^2 = 16$

  This is a circle with centre (2, 1) and radius 4.



- Substitute in $z = x + yi$ and rearrange to get:

  $2|x + yi| = x + yi + x - yi + 4$

  $2\sqrt{x^2 + y^2} = 2x + 4$

  $\sqrt{x^2 + y^2} = x + 2$

  $x^2 + y^2 = (x + 2)^2$

  $x^2 + y^2 = x^2 + 4x + 4$

  $x = \frac{1}{4}y^2 - 1$

This is a parabola, but instead of the parabolas we're used to with $y = ax^2 + bx + c$, the x's and y's are switched! This results in a parabola which is flipped on its side, with a stretch factor of $\frac{1}{4}$ and a horizontal shift to the left 1.