Differentiation
ANSWERS

Differentiation Basics

Functional Notation
STOP AND CHECK (PAGE 5)

- For this first one, we can split it into two functions, one we will call u and the other will call v:
  - Let u = $x^2 + 2$
  - let v = $\cos(x^2 + 2)$
  - Therefore, y = v(u).

We say that y is equal to v of u.

- For the second one we will use functions as our notation, but there are 3 functions here, so we create 3 functions: g, h, j. Remember that the letters aren't important.
  - Let $g(x) = 2x$
  - Let $h(g(x)) = \ln | g(x) |$
  - Let $j(h(g(x))) = (h(g(x)))4$
  - $f(x) = j(h(g(x))$

  This may seem very confusing and overly complicated, but it makes our lives a lot easier when it comes to applying differentiation rules like the chain rule. We've included this at the start because the notation is super important.

The Derivative and Tangent Lines
STOP AND CHECK (PAGE 7)

- This is a great question because it gets us closer to what it means to take a derivative and why finding the value of the derivative at a point gives us the gradient. First of all, we all know that:

$$y = x^2$$
$$\frac{dy}{dx} = 2x$$

Plugging in a point like x = 2 we find that the gradient is equal to 4. Now let's see what happens when we find the average gradient between two points around x = 2 for our original curve. We'll start with two points like x = 2 and x = 2.5, pretty far away. The gradient between these two points is:

$$\frac{rise}{run} = \frac{(x+h)^2 - x^2}{h}$$

Our h value is 0.5 because that's how far away the point is. So we plug in our points:

$$\frac{2.5^2 - 2^2}{0.5} = 4.5$$

Not a bad estimate, but not perfect since we know the actual gradient is 4. Now taking a closer point, let's do x = 2 and x = 2.1, here we get:

$$\frac{2.1^2 - 2^2}{0.1} = 4.1$$

We are getting closer! Now, let's take two points that are really close together, like x = 2 and x = 2.001. This leaves us with the gradient of this line being 4.001, even closer to our derivative!

What this tells us is that if we keep picking points closer and closer to each other, our gradient will get more and more accurate until it's eventually perfect! In fact, this is the basis of calculus and is pretty crucial to how derivatives work.

- The crucial connection is that the gradient of a curve at a point as given by its derivative is also the gradient of the tangent to that point on the curve. Wow. Now, since m is the gradient and we have that already, the only thing we would have to do to find the equation of the tangent is plug in a value of x and a value of y to find c. We know how to find these since we have the original function (so even if we only have x, we can plug it into our original equation to find y.

Finding Derivatives from Level 2

STOP AND CHECK (PAGE 9)

- $f(x) = \frac{4}{x^{4}}$

  We should rewrite this as:

  $$f'(x) = 4x^{-4}$$

  Now take the derivative as normal, multiply the coefficient by the power and decrease the power by one.

  $$f'(x) = -16x^{-5}$$

- $f(x) = \sqrt[3]{x^{2}}$

  Again, rewriting this:

  $$f(x) = x^{2/3}$$

  Now we take the derivative:

  $$f'(x) = \frac{2}{3}x^{-1/3}$$

Differentiation Basics

QUICK QUESTIONS (PAGE 9)

- $f'(x) = \frac{-6}{x^{4}}$

  $$f'(1) = \frac{-6}{1^{4}}$$

  $$f'(1) = -6$$

- $f'(x) = \frac{1}{5}\sqrt[5]{\frac{2}{x^{4}}}$

  $$f'(1) = \frac{1}{5}\sqrt[5]{\frac{2}{1^{4}}}$$

$$f'(1) = \sqrt[5]{\frac{2}{5}}$$

- $f'(x) = 4x + 4$

$$-3 = 4x + 4$$
$$-7 = 4x$$
$$x = \frac{-7}{4}$$

New Functions to Differentiate

Trig Functions

STOP AND CHECK (PAGE 11)

- On the formula sheet! You should get very familiar with the formula sheet, it's your best friend in an exam.

Exponentials

STOP AND CHECK (PAGE 11)

- The rule is simply that the derivative of e^x is e^x.

Logarithmic Functions

STOP AND CHECK (PAGE 13)

- Another name for $\log_e(x)$ is $\ln(x)$.
- The derivative of $\ln(x)$ is $\frac{1}{x}$

New Functions to Differentiate

QUICK QUESTIONS (PAGE 13)

- The first derivative of $\sin(x)$ is $\cos(x)$, and if we differentiate again we get $-\sin(x)$. So the second derivative of $\sin(x)$ is $-\sin(x)$.

- Since it doesn't matter how many times we differentiate e^x, we'll still end up with e^x!
- $f'(x) = \frac{1}{x}$
  - $f''(x) = \frac{-1}{x^2}$
  - $f'''(x) = \frac{2}{x^3}$

Differentiation Rules

The Chain Rule

STOP AND CHECK (PAGE 16)

- Let $u = 2x^2 - 3x + 4$

  So then $y = \sqrt{u}$

$$\frac{du}{dx} = 4x - 3$$
$$\frac{dy}{du} =- \frac{1}{2\sqrt{u}}$$

So now we have collected all the derivatives, we can plug them into the chain rule:

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} =- \frac{4x-3}{2\sqrt{u}}$$

And finally, we have to make sure to substitute back in for u:

$$\frac{dy}{dx} =- \frac{4x-3}{2\sqrt{2x^2-3x+4}}$$

- Let $u = 6x^3 + 4x + 1$

  So then $y = u^3$

$$\frac{du}{dx} = 18x^2 + 4$$
$$\frac{dy}{du} = 3u^2$$

Plug everything into the chain rule to get:

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 3u^2(18x^2 + 4)$$

And substitute back in for u to get our final answer:

$$\frac{dy}{dx} = 3(6x^3 + 4x + 1)^2(18x^2 + 4)$$

- This one's a bit more complicated, but we can use some sneaky algebra tricks to make it more approachable.

$$y = \frac{3}{\sqrt[4]{(x+2)^3}} = \frac{3}{(x+2)^{3/4}} = 3(x + 2)^{-3/4}$$

Alright, that doesn't look super lovely but at least now there's a clearer choice of u.

Let $u = x + 2$

Then $y = 3u^{-3/4}$

$$\frac{du}{dx} = 1$$
$$\frac{dy}{du} = -\frac{9}{4}u^{-7/4}$$

Which we plug into the chain rule to get:

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = -\frac{9}{4}u^{-7/4}$$

And then put u back in and clean up (reversing our sneaky algebra tricks) to get the final answer of:

$$\frac{dy}{dx} = -\frac{9}{4\sqrt[4]{(x+2)^7}}$$

The Product Rule

STOP AND CHECK (PAGE 18)

- Let $u = \frac{1}{x^3} = x^{-3}$

  Let $v = \sqrt{x} = x^{1/2}$

$$\frac{du}{dx} = -3x^{-4}$$
$$\frac{dv}{dx} = \frac{1}{2}x^{-½}$$

Then plug everything into the product rule formula to get:

$$\frac{dy}{dx} = \frac{1}{2}x^{-3}x^{-1/2} - 3x^{-4}x^{1/2}$$
$$= \frac{1}{2x^3\sqrt{x}} - \frac{3}{x^4}\sqrt{x}$$

- Let $u = x^2$

  Let v = 2 + 7x

$$\frac{du}{dx} = 2x$$
$$\frac{dv}{dx} = 7$$

  Then plug everything into the product rule formula to get:

$$\frac{dy}{dx} = 7x^2 + 2x(2 + 7x)$$

- Let $u = \sqrt[4]{x} = x^{1/4}$

  Let v = x^3 + 9

$$\frac{du}{dx} = \frac{1}{4}x^{-3/4} = \frac{1}{4\sqrt[4]{x^3}}$$
$$\frac{dv}{dx} = 3x^2$$

  Then plug everything into the product rule formula to get:

$$\frac{dy}{dx} = 3x^2(\sqrt[4]{x}) + \frac{x^3+9}{4\sqrt[4]{x^3}}$$

The Quotient Rule

STOP AND CHECK (PAGE 19)

- $y' = \frac{6x^2 - 6x(2x-4)}{9x^4}$

- $y' = \frac{2}{3x^{2}} - \frac{4x-8}{3x^{3}}$

- $y' = \frac{\frac{7x-2}{2\sqrt{x}} - 7\sqrt{x}}{(7x-2)^{2}}$
  - $y' = \frac{1}{2\sqrt{x}(7x-2)} - \frac{7\sqrt{x}}{(7x-2)^{2}}$
- $y' = \frac{-2(\frac{-1}{3\sqrt[3]{x^{2}}}}{(4-\sqrt[3]{x})^{2}}$
  - $y' = \frac{2}{27\sqrt{x^{2}}(4-\sqrt[3]{x})^{2}}$

Parametric Differentiation

STOP AND CHECK (PAGE 21)

- $\frac{dx}{dt} = 4t$
  - $\frac{dy}{dt} = 1$
  - $\frac{dy}{dx} = \frac{dy}{dt}$
  - $\frac{dt}{dx} = \frac{1}{4t}$
- $\frac{dx}{dt} = \frac{1}{2\sqrt{t}}$
  - $\frac{dy}{dt} = 3t^2$
  - $\frac{dy}{dx} = \frac{dy}{dt}$
  - $\frac{dt}{dx} = 6t^2\sqrt{t}$
- $\frac{dx}{dt} = e^t$
  - $\frac{dy}{dt} = \frac{-1}{t^{2}}$
  - $\frac{dy}{dx} = \frac{dy}{dt}$
  - $\frac{dt}{dx} = \frac{-1}{t^{2}e^{t}}$

Differentiation Rules

QUICK QUESTIONS (PAGE 21)

- Here we'll use the chain rule:
  - Let $u = 5x + 2$
  - So $y = \cos(u)$
  - $\frac{du}{dx} = 5$

- $\frac{dy}{du}$ = -sin(u)
- So $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$ = -5sin(u)
- Substitute back in for u to get:

$$\frac{dy}{dx} = -5\sin(5x + 2)$$

- Again, we'll use the chain rule:
  - Let u = x^4
  - So y = ln(u)
  - $\frac{du}{dx}$ = 4x^3
  - $\frac{dy}{du} = \frac{1}{u}$
  - So $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{4x^3}{x^4} = \frac{4}{x}$

- This question is multiplying two functions together, so we'll use the product rule. We can use our answers from the previous questions to speed things along too.
  - Let u = cos$(5x + 2)$
  - Let v = ln(x^4)
  - $\frac{du}{dx}$= -5sin$(5x + 2)$
  - $\frac{dv}{dx} = \frac{4}{x}$
  - Now just plug into the product rule to get:

$$\frac{dy}{dx} = \frac{4cos(5x+2)}{x} - 5ln(x^4)sin(5x + 2)$$

- We'll need to use the quotient rule for this equation since there's a fraction with functions as both the numerator and denominator. It'll also be a good idea to use the chain rule to differentiate the numerator and denominator of the function.
  - Let u = cos(x^2)
  - Let v = ln(x^2)
  - $\frac{du}{dx}$ = -2$x$sin(x^2)(by using the chain rule)
  - $\frac{dv}{dx} = \frac{2x}{x^2} = \frac{2}{x}$ (by using the chain rule)
  - Then we plug everything into the quotient rule to get the final answer:

$$\frac{dy}{dx} = \frac{-2xsin(x^2)ln(x^2) - \frac{2}{x}cos(x^2)}{(ln(x^2))^2}$$

- $\frac{dx}{dt} = \frac{2}{t}$
  - $\frac{dy}{dt}$ = -sin(t) + 15t^2
  - $\frac{dy}{dx} = \frac{dy}{dt}$
  - $\frac{dt}{dx} = \frac{-tsin(t) + 15t^3}{2}$

Interpreting Features of Graphs

Maxima, Minima and Tangent Lines

STOP AND CHECK (PAGE 22)

- $\frac{dy}{dx}$ = 2x + 5 = 0
  - x = -2.5
  - y = (-2.5)^2 + 5(-2.5) + 2
  - y = -4.25
  - y = $\frac{-17}{4}$
- $\frac{dy}{dx}$ = -4x + 3 = 0
  - x = 0.75
  - y = -2(0.75)^2 + 3(0.75) - 6
  - y = -4.875
  - y = $\frac{-39}{8}$
- $\frac{dy}{dx}$ = 3x^2 + 4x + 5 = 0

  This quadratic has no real roots, so the function has no real maxima or minima.
- $\frac{dy}{dx}$ = 1 - $\frac{10}{x^3}$
  - x = $\sqrt[3]{10}$
  - y = $\frac{(\sqrt[3]{10})^3 + 5}{(\sqrt[3]{10})^2}$
  - y = $\frac{15}{(\sqrt[3]{10})^2}$
  - y = 3.232
  - $\frac{d^2y}{dx^2} = \frac{30}{x^4} = \frac{30}{\sqrt[3]{10}^4}$ = 1.392 > 0

So, y = 3.232 is a minimum.

- $\frac{dy}{dx} = 2x + 2$
    - $\frac{dy}{dx} = 2(-2) + 2$
    - $\frac{dy}{dx} = -2$

  Normal gradient: $\frac{-1}{-2} = \frac{1}{2}$

  When $x = -2$:
    - $y = (-2)^2 + 2(-2) - 4$
    - $y = -4$
    - $y - y_1 = m(x - x_1)$
    - $y - (-4) = \frac{1}{2}(x - (-2))$
    - $y = \frac{x}{2} - 3$ (this is the final equation for the normal)

Undefined Functions

STOP AND CHECK (PAGE 23)

- The function has no y value for that given *x* value. The function is undefined at a hole.
- At $x = -2$, the denominator becomes $-2 + 2 = 0$. This means the function requires dividing by zero, which is undefined.

Continuity

STOP AND CHECK (PAGE 24)

- A function is continuous if we can draw it without lifting the pen off the page.
- An example of a continuous function is a parabola. An example of a discontinuous function is the one given on page 23.

Limits

STOP AND CHECK (PAGE 25)

- $2(-3) - 6 = -6 - 6$
    - $= -12$
- $\frac{3}{6-6} = \frac{3}{0}$

    - This is undefined so the limit does not exist (Mean Girls scream in the distance)
- As x gets very big (goes to infinity), the −6 on the numerator and +7 on the denominator become insignificant. So the limit is $\frac{2}{4}$ which is 0.5.

Concavity and the Second Derivative

STOP AND CHECK (PAGE 27)

- First we find the derivatives. For $f(x) = x^3 - 4x^2$ this is:
    - $f'(x) = 3x^2 - 8x$
    - $f''(x) = 6x - 8$

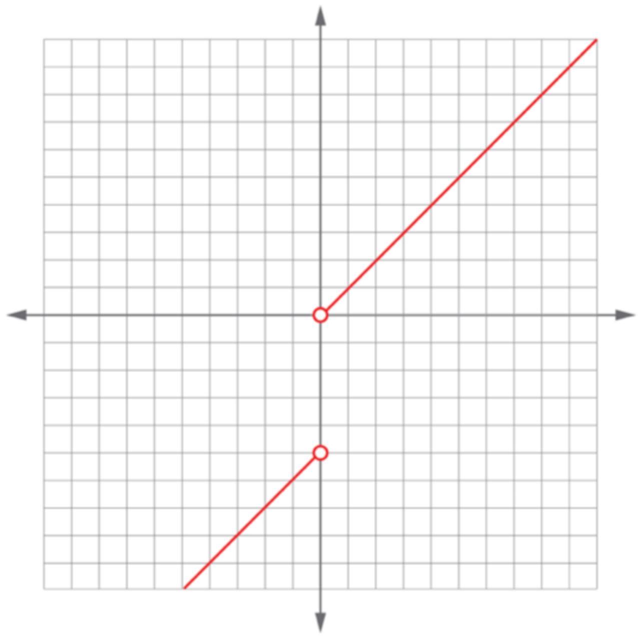
  The second derivative is positive when $6x - 8 > 0$ which is when $x > \frac{8}{6}$ So it is negative when $x < \frac{8}{6}$
- Now for $y = (x + 1)^2(x - 2)$ the derivative is:
    - $y' = 3$ (by expanding the brackets)
    - $y'' = 6x$

  The second derivative is positive when $x > 0$ and negative when $x < 0$.

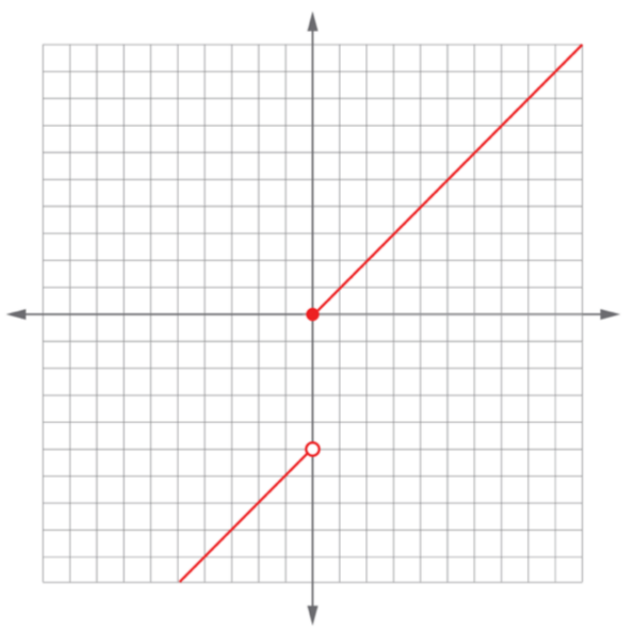
Interpreting Features of Graphs

QUICK QUESTIONS (PAGE 27)

- The limit of $f(x)$ is not defined at $x = 0$ because from the left, it looks like the limit will be 5, but from the right, it looked like the limit will be 0.



- The function is defined at $x = 0$. $f(0) = 0$.
  - Note that the limit is still not defined at $x = 0$, since the two sides still look like they will disagree as they get closer to $x = 0$. So, a function can be defined at a point and still not have a limit there!



Applications of Derivatives

Optimisation

STOP CHECK (PAGE 30)

- We want to find the maximum area of a rectangle inside a circle of radius 2. First we would draw a diagram of the situation, in this case we will draw the circle at the centre of the x-y plane (with a radius of 2).
  The rectangle has a width of 2x and a height of 2y, so the area is given by:

  $$\text{Area} = \text{base} \times \text{height}$$
  $$\text{Area} = 2x2y$$
  $$\text{Area} = 4xy$$

  We can't differentiate this directly since we've got two variables. This is a bit of a tricky step, but triangles within a semicircle have a right angled triangle with the diameter as the hypotenuse, so we can apply Pythagoras to get:

  $$x^2 + y^2 = 4^2$$

  Therefore, $y = \sqrt{16 - x^2}$
  Which we can substitute into our area equation to find:

  $$A = 4x\sqrt{16 - x^2}$$

  We just need to differentiate this:

  $A' = 4\sqrt{16 - x^2} - \frac{4x^2}{\sqrt{16-x^2}}$ (by using the chain rule and the product rule)

  This is a maximum when A' = 0. So:
  - $0 = 4\sqrt{16 - x^2} - \frac{4x^2}{\sqrt{16-x^2}}$
  - $0 = 4(16 - x^2) - 4x^2$
  - $0 = 64 - 4x^2 - 4x^2$
  - $0 = -8(x^2 - 8)$
  - $0 = -8(x + \sqrt{8})(x - \sqrt{8})$

- So $x = \sqrt{8}$ and $-\sqrt{8}$

It doesn't make sense to have a negative length, so we choose $x = \sqrt{8}$ and put that into our area equation to get an area of 32 (and notice that $y = \sqrt{8}$ as well).

- Again, we start by figuring out how to express this situation mathematically. Let's call the wires a and b. Let the position where the wire meets the ground be x. The length of the wire is given by the equation:

$$\text{Length} = a + b$$

Now we express a and b as functions of x. We notice that they form two right angled triangles with the hypotenuse being a and b, the heights being 15 and 6m, and the widths being x and $(20 - x)$ respectively. Now we express a and b:

$$a^2 = 15^2 + x^2$$

So, $a = \sqrt{15^2 + x^2}$

$$b^2 = 6^2 + (20 - x)^2$$

So, $b = \sqrt{6^2 + (20 - x)^2}$

We can now express the length as:

$$L = \sqrt{15^2 + x^2} + \sqrt{6^2 + (20 - x)^2}$$

The hard part is over! We differentiate using the product rule and chain rule to get:

$$L' = \frac{-x}{\sqrt{15^2 + x^2}} - \frac{x - 40}{\sqrt{6^2 + (20 - x)^2}}$$

We have omitted some simplification steps here. Now we set it equal to zero, since the length is at a minimum when the derivative is 0. First, there is an obvious (but incorrect answer), which is when x = 0. The other, we must find algebraically.

$$0 = \frac{-x}{\sqrt{15^2 + x^2}} - \frac{-x}{\sqrt{6^2 + (20-x)^2}}$$

$$0 = x\sqrt{15^2 + x^2} = x\sqrt{6^2 + (20-x)^2}$$

Now cancel the *x*'s by squaring both sides:

$$15^2 + x^2 = 6^2 + (20 - x)^2$$

$$40x = 62 + 202 - 152$$

$$X = 5.275$$

Which is our final answer. The wire should meet 5.275m away from the first pole for the wires to have a minimum length.

Related Rates of Change

STOP CHECK (PAGE 33)

- We are given the derivative $\frac{dp}{dt}$ = 100 and the rate of change of food consumption is the derivative $\frac{df}{dp}$ = 2p
  To find the rate of change of food over time we need the derivative:

  $$\frac{df}{dt} = \frac{df}{dp} \times \frac{dp}{dt}$$

  We have both of these, so:

  $$\frac{df}{dt} = 200p$$

  We would need the equation for p(t) to find this in terms of t.

- The derivative we are given is $\frac{dx}{dt} = 2$
  The derivative we get from the area equation is A = $6x^2$, is $\frac{dA}{dx}$ = 12*x.* We want the derivative $\frac{dA}{dt} = \frac{dA}{dx} \times \frac{dx}{dt}$
  Substituting the derivatives we get:

  $$\frac{dA}{dt} = 24x$$

When the side length is 10, the rate of change is 240cms^{-1}.

- The derivative we are given is $\frac{dV}{dt}$ = 2, using the volume of a cone, we need a derivative of V = $\pi r^2 \frac{d}{3}$ where d is the depth of the cone.
  We first need the volume in terms of r or d only. This is found by the relationship between the side lengths, which is:

  $$\frac{d}{r} = \frac{15}{5}$$

  So d = 3r
  Now, the derivative is:

  $$\frac{dV}{dr} = 3\pi r^2 \text{ or } \frac{dV}{dd} = \pi \frac{d^2}{9}$$

  Which we then used to find the desired derivatives $\frac{dd}{dt}$ and $\frac{dr}{dt}$.

