STUDY TIME

LEVEL 3 CALCULUS

DIFFERENTIATION

NCEA Answer Workbook

Section One
The Foundations

1. Simple Differentiation

a. A function takes the input of one or more values, does some stuff to the input, and then outputs a single value. We often call the input x and the output y, but they can be any variables. Functions can have many forms. They can be polynomial, logarithmic, exponential, trigonometric, or others.

b. $\frac{dy}{dx}$, y', $f'(x)$ These all mean the same thing!

c. *i.* $y = 5x^3$

$\frac{dy}{dx} = 15x^2$

ii. $y = 8x^5$

$\frac{dy}{dx} = 40x^4$

iii. $y = \frac{4}{6}x^6$

$\frac{dy}{dx} = 4x^5$

iv. $y = 5e^x$

$\frac{dy}{dx} = 5e^x$

v. $f(x) = 3x^{-2}$

$f'(x) = -6x^{-3}$

vi. $y = e^{4x}$

$\frac{dy}{dx} = 4e^{4x}$

vii. $y = \ln(x)$

$\frac{dy}{dx} = \frac{1}{x}$

viii. $y = 12x^2$

$\frac{dy}{dx} = 24x$

ix. $f(x) = \sin(x)$

$f'(x) = \cos(x)$

x. $y = \tan(x)$

$\frac{dy}{dx} = \sec^2(x)$

xi. $f(x) = 9e^{3x}$

$f'(x) = 27e^{3x}$

xii. $y = 7$

$\frac{dy}{dx} = 0$

xiii. $y = 6\cos(x)$

$\frac{dy}{dx} = -6\sin(x)$

xiv. $y = \frac{4}{6}\ln(x)$

$\frac{dy}{dx} = \frac{4}{6x}$

xv. $6x^{-1}$

$\frac{dy}{dx} = -6x^{-2}$

xvi. $y = \frac{5}{2}\operatorname{cosec}(x)$

$\frac{dy}{dx} = -\frac{5}{2}\operatorname{cosec}(x)\cot(x)$

xvii. $h = 15\sin(\theta)$

$\frac{dh}{d\theta} = 15\cos(\theta)$

xviii. $y = -12e^{-2x}$

$\frac{dy}{dx} = 24e^{-2x}$

xix. $f(x) = -4x^4$

$f'(x) = -16x^3$

xx. $y = \frac{1}{4}e^{-8x}$

$\frac{dy}{dx} = -2e^{-8x}$

xxi. $y = -\frac{5}{3}\ln(x)$

$\frac{dy}{dx} = -\frac{5}{3x}$

d. *i.* $y = -6ax$

$\frac{dy}{dx} = -6a$

ii. $y = 3bcx^{-5}$

$\frac{dy}{dx} = -15bcx^{-6}$

iii. $y = e^{bx}$

$\frac{dy}{dx} = be^{bx}$

iv. $y = \frac{5}{6}ax^3$

$\frac{dy}{dx} = \frac{5}{2}ax^2$

v. $y = -\frac{\ln(x)}{a}$

$\frac{dy}{dx} = -\frac{1}{ax}$

vi. $y = -ab$

$\frac{dy}{dx} = 0$

vii. $y = -be^{-ax}$

$\frac{dy}{dx} = abe^{-ax}$

viii. $y = b\tan(x)$

$\frac{dy}{dx} = b\sec^2(x)$

ix. $y = bx$

$\frac{dy}{dx} = b$

x. $y = \frac{3c}{2}\ln(x)$

$\frac{dy}{dx} = \frac{3c}{2x}$

xi. $y = \frac{a}{b}\cos(x)$

$\frac{dy}{dx} = -\frac{a}{b}\sin(x)$

xii. $y = e^{bcx}$

$\frac{dy}{dx} = bce^{bcx}$

xiii. $y = c^3$

$\frac{dy}{dx} = 0$

xiv. $y = b^2e^{cx}$

$\frac{dy}{dx} = b^2ce^{cx}$

xv. $y = 5a^2b^3x^4$

$\frac{dy}{dx} = 20a^2b^3x^3$

e. *i.* $\frac{dy}{dx} = -6x^2$

$\frac{d^2y}{dx^2} = -12x$

ii. Note: In older versions of the workbook this question is a repeat of *i.*. The replacement question is provided here in blue.

$y = -5x^5$

$\frac{dy}{dx} = 25x^4$

$\frac{d^2y}{dx^2} = 100x^3$

iii. $y = 3e^{4x}$

$\frac{dy}{dx} = 12e^{4x}$

$\frac{d^2y}{dx^2} = 48e^{4x}$

iv. $y = 8\ln(x)$

$\frac{dy}{dx} = 8x^{-1}$

$\frac{d^2y}{dx^2} = -8x^{-2}$

v. $y = 3x^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{3}{2}x^{-\frac{1}{2}}$

$\frac{d^2y}{dx^2} = -\frac{3}{4}x^{-\frac{3}{2}}$

vi. $y = 5\sin(x)$

$\frac{dy}{dx} = 5\cos(x)$

$\frac{d^2y}{dx^2} = -5\sin(x)$

2. Fractional Exponents and Simplifying Derivatives

a. *i.* $a^n \times a^m = a^{n+m}$

ii. $(a^n)^m = a^{nm} = a^{mn}$

iii. $\frac{a^m}{a^n} = a^{m-n}$

iv. $\frac{1}{a^n} = a^{-n}$

v. $a^{\frac{1}{n}} = \sqrt[n]{a}$

vi. $a^{\frac{n}{m}} = \sqrt[m]{a^n}$

vii. $a^1 = a$

viii. $a^0 = 1$

b. *i.* $(\frac{3a^2}{2c^3})^3$

$\frac{3^3a^6}{2^3c^9} = \frac{27a^6}{8c^9}$

ii. $(2a^3)^{-3}$

$(\frac{1}{2a^3})^3 = \frac{1}{8a^9}$

iii. $\sqrt{\frac{121c^2}{a^2b^2}}$

$(\frac{121c^2}{a^2b^2})^{\frac{1}{2}} = \frac{11c}{ab}$

iv. $(\frac{16a^2}{a^5})^{-2}$

$(\frac{a^5}{16a^2})^2 = \frac{a^{10}}{16^2a^4} = \frac{a^{10}}{256a^4}$

v. $(4c^{-3})^{\frac{1}{2}}$

$(\frac{4}{c^3})^{\frac{1}{2}} = \frac{2}{c^{\frac{3}{2}}}$

vi. $\sqrt{(\frac{16x^3}{c^2})}$

$(\frac{16x^3}{c^2})^{\frac{1}{2}} = \frac{4x^{\frac{3}{2}}}{c}$

vii. $((x+2)(x+2))^{\frac{1}{2}}$

$((x+2)^2)^{\frac{1}{2}} = x+2$

viii. $\sqrt{(\frac{32}{x^5})^3}$

$(\frac{32}{x^5})^{\frac{3}{2}} = \frac{32^{\frac{3}{2}}}{x^{\frac{15}{2}}}$

c. *i.* $f(x) = (4x^3)^{\frac{1}{2}}$

Start by simplifying:

$f(x) = 2x^{\frac{3}{2}}$

Then differentiate:

$f'(x) = 3x^{\frac{1}{2}} = 3\sqrt{x}$

ii. $y = (x^{\frac{3}{2}})^3$

Start by simplifying:

$y = x^{\frac{9}{2}}$

Then differentiate:

$\frac{dy}{dx} = \frac{9}{2}\, x^{\frac{7}{2}}$

iii. $y = \sqrt{(\frac{4x^2}{x^2})^3}$

Start by simplifying:

$y = \sqrt{(4)^3}$

$y = \sqrt{64}$

$y = 8$

Then differentiating:

$\frac{dy}{dx} = 0$

iv. $y = \frac{3}{x^3}$

We start by writing this in a way that is easier to differentiate:

$y = 3x^{-3}$

Then differentiate:

$\frac{dy}{dx} = -9x^{-4} = -\frac{9}{x^4}$

v. $f(x) = (\frac{1}{x^3})^{-2}$

Start by simplifying:

$f(x) = (x^3)^2 = x^6$

Then differentiate:

$f'(x) = 6x^5$

vi. $f(x) = \sqrt{x}$

Start by writing with fractional indices:

$f(x) = x^{\frac{1}{2}}$

Then differentiate:

$f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

3. Sum Rule

a. *i.* To calculate the derivative of a function that is the sum of two functions, e.g. $y = 5x^3 + \cos(x)$.

ii. $\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$

Basically, we differentiate the two functions with respect to x separately, then add them together.

iii. Step 1: Differentiate u with respect to x, to give $\frac{du}{dx}$

Step 2: Differentiate v with respect to x, to give $\frac{dv}{dx}$

Step 3: Add these derivatives to get $\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$

b. i. $y = 4x^3 + 2x^5$

$\frac{dy}{dx} = 12x^2 + 10x^4$

ii. $y = \sin(x) + \tan(x)$

$\frac{dy}{dx} = \cos(x) + \sec^2(x)$

iii. $y = 5x^2 - 4x - 3$

$\frac{dy}{dx} = 10x - 4$

iv. $y = 3x^4 + 7x^2 - 10x + 2$

$\frac{dy}{dx} = 12x^3 + 14x - 10$

v. $y = 9\ln(x) + 3\cos(x)$

$\frac{dy}{dx} = \frac{9}{x} - 3\sin(x)$

vi. $y = 3e^{9x} - 12x^3$

$\frac{dy}{dx} = 27e^{9x} - 36x^2$

vii. $y = 3\ln(x) - 4\sec(x)$

$\frac{dy}{dx} = \frac{3}{x} - 4\sec(x)\tan(x)$

viii. $y = \cos(x) - 6e^{-6x}$

$\frac{dy}{dx} = -\sin(x) + 36e^{6x}$

ix. $y = 6x^{-3} - 3x^{-2}$

$\frac{dy}{dx} = -18x^{-4} + 6x^{-3}$

x. $y = -4e^{-2x} + \ln(x)$

$\frac{dy}{dx} = 8e^{-2x} + \frac{1}{x}$

4. Chain Rule

a. i. To calculate the derivative of a composite function, which is an equation that applies one function to the results of another, eg. $y = \cos(e^{3x})$.

ii. $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

This means the derivative of y with respect to x is calculated from the derivative of y with respect to u multiplied by the derivative of u with respect to x.

iii. Step 1: Differentiate y with respect to u, to give $\frac{dy}{du}$

Step 2: Differentiate u with respect to x, to give $\frac{du}{dx}$

Step 3: Multiply these derivatives together $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

Step 4: Substitute in $u = f(x)$ where u appears in your final answer, so the answer is only in terms of x.

b. *i.* $y = (4x)^3$

Let $u = 4x$

So $y = u^3$

Differentiate both functions:

$\frac{dy}{du} = 3u^2$

$\frac{du}{dx} = 4$

Apply the chain rule:

$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

$\frac{dy}{dx} = 12u^2$

Substitute back in for u and simplify:

$\frac{dy}{dx} = 12(4x)^2$

$\frac{dy}{dx} = 12(4x)^2$

$\frac{dy}{dx} = 192x^2$

ii. $y = \sin(5x)$

Let $u = 5x$

So $y = \sin(u)$

Differentiate both functions:

$\frac{dy}{du} = \cos(u)$

$\frac{du}{dx} = 5$

Apply the chain rule:

$\frac{dy}{dx} = \frac{dy}{du}\frac{dy}{dx}$

$\frac{dy}{dx} = 5\cos(u)$

Substitute back in for u and simplify:

$\frac{dy}{dx} = 5\cos(5x)$

iii. $y = (5x^2)^4$

Let $u = 5x^2$

So $y = u^4$

Differentiate both functions:

$\frac{dy}{du} = 4u^3$

$\frac{du}{dx} = 10x$

Apply the chain rule:

$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

$\frac{dy}{dx} = 40xu^3$

Substitute back in for u and simplify:

$\frac{dy}{dx} = 40x(5x^2)^3$

$\frac{dy}{dx} = 200x^7$

iv. $y = \ln(6x^4)$

Let $u = 6x^4$

So $y = \ln(u)$

Differentiate both functions:

$\frac{dy}{du} = \frac{1}{u}$

$\frac{du}{dx} = 24x^3$

Apply the chain rule:

$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

$\frac{dy}{dx} = \frac{24x^3}{u}$

Substitute back in for u and simplify:

$\frac{dy}{dx} = \frac{24x^3}{6x^4}$

$\frac{dy}{dx} = \frac{4}{x}$

v. $y = \cos(\sin(x))$

Let $u = \sin(x)$

So $y = \cos(u)$

Differentiate both functions:

$\frac{dy}{du} = -\sin(u)$

$\frac{du}{dx} = \cos(x)$

Apply the chain rule:

$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

$\frac{dy}{dx} = -\sin(u)\cos(x)$

Substitute back in for u:

$\frac{dy}{dx} = -\sin(\sin(x))\cos(x)$

vi. $y = \tan(e^{6x})$

Let $u = e^{6x}$

So $y = \tan(u)$

Differentiate both functions:

$\frac{dy}{du} = \sec^2(u)$

$\frac{du}{dx} = 6e^{6x}$

Apply the chain rule:

$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

$\frac{dy}{dx} = 6\sec^2(u)e^{6x}$

Substitute back in for u:

$\frac{dy}{dx} = 6\sec^2(e^{6x})e^{6x}$

vii. $y = 4e^{7x^5}$

Let $u = 7x^5$

So $y = 4e^u$

Differentiate both functions:

$\frac{dy}{du} = 4e^u$

$\frac{du}{dx} = 35x^4$

Apply the chain rule:

$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

$\frac{dy}{dx} = 140x^4e^u$

Substitute back in for u:

$\frac{dy}{dx} = 140x^4e^{7x^5}$

viii. $y = \frac{5}{2}\cos^3(x)$

Let $u = \cos(x)$

So $y = \frac{5}{2}u^3$

Differentiate both functions:

$\frac{dy}{du} = \frac{15}{2}u^2$

$\frac{du}{dx} = -\sin(x)$

Apply the chain rule:

$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

$\frac{dy}{dx} = -\frac{15}{2}u^2\sin(x)$

Substitute back in for u:

$\frac{dy}{dx} = -\frac{15}{2}\cos^2(x)\sin(x)$

ix. $y = 6e^{7\sin(x)}$

Let $u = 7\sin(x)$

So $y = 6e^{u}$

Differentiate both functions:

$\frac{dy}{du} = 6e^{u}$

$\frac{du}{dx} = 7\cos(x)$

Apply the chain rule:

$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

$\frac{dy}{dx} = 42e^{u}\cos(x)$

Substitute back in for u:

$\frac{dy}{dx} = 42e^{7\sin(x)}\cos(x)$

x. $y = \sin(7\ln(x))$

Let $u = 7\ln(x)$

So $y = \sin(u)$

Differentiate both functions:

$\frac{dy}{du} = \cos(u)$

$\frac{du}{dx} = \frac{7}{x}$

Apply the chain rule:

$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

$\frac{dy}{dx} = \frac{7\cos(u)}{x}$

Substitute back in for u:

$\frac{dy}{dx} = \frac{7\cos(7\ln(x))}{x}$

5. Product Rule

a. *i.* To calculate the derivative of a function that is the product of two functions. Or in other words, one function of x multiplied by another, eg. $y = x^2\sin(x)$.

ii. $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

This means the derivative of y with respect to x is calculated by multiplying u by the derivative of v, then multiplying v by the derivative of u, and adding those together.

iii. Step 1: Differentiate u with respect to x, to get $\frac{du}{dx}$

Step 2: Differentiate v with respect to x, to get $\frac{dv}{dx}$

Step 3: Apply the product rule to get $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

b. *i.* $y = x^4\sin(x)$

Let $u = x^4$

Let $v = \sin(x)$

Differentiate both functions:

$\frac{du}{dx} = 4x^3$

$\frac{dv}{dx} = \cos(x)$

Apply the product rule:

$\frac{dy}{dx} = u\frac{dv}{dx}+v\frac{du}{dx}$

$\frac{dy}{dx} = x^4\cos(x) + 4x^3\sin(x)$

ii. $y = e^{2x}\ln(x)$

Let $u = e^{2x}$

Let $v = \ln(x)$

Differentiate both functions:

$\frac{du}{dx} = 2e^{2x}$

$\frac{dv}{dx} = \frac{1}{x}$

Apply the product rule:

$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$\frac{dy}{dx} = \frac{e^{2x}}{x} + 2e^{2x}\ln(x)$

iii. $y = x^2e^{-2x}$

Let $u = x^2$

Let $v = e^{-2x}$

Differentiate both functions:

$\frac{du}{dx} = 2x$

$\frac{dv}{dx} = -2e^{-2x}$

Apply the product rule:

$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$\frac{dy}{dx} = -2x^2e^{-2x} + 2xe^{-2x}$

iv. $y = \tan(x)\sin(x)$

Let $u = \tan(x)$

Let $v = \sin(x)$

Differentiate both functions:

$\frac{du}{dx} = \sec^2(x)$

$\frac{dv}{dx} = \cos(x)$

Apply the product rule:

$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$\frac{dy}{dx} = \tan(x)\cos(x) + \sin(x)\sec^2(x)$

v. $y = 5\ln(x)\cos(x)$

Let $u = 5\ln(x)$

Let $v = \cos(x)$

Differentiate both functions:

$\frac{du}{dx} = \frac{5}{x}$

$\frac{dv}{dx} = -\sin(x)$

Apply the product rule:

$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$\frac{dy}{dx} = -5\ln(x)\sin(x) + \frac{5\cos(x)}{x}$

vi. $y = -\frac{3}{4}x^8e^{-3x}$

Let $u = -\frac{3}{4}x^8$

Let $v = e^{-3x}$

Differentiate both functions:

$\frac{du}{dx} = -6x^7$

$\frac{dv}{dx} = -3e^{-3x}$

Apply the product rule:

$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$\frac{dy}{dx} = \frac{9}{4}x^8e^{-3x} - 6x^7e^{-3x}$

vii. $y = 4\sin(x)\cos(x)$

Let $u = 4\sin(x)$

Let $v = \cos(x)$

Differentiate both functions:

$\frac{du}{dx} = 4\cos(x)$

$\frac{dv}{dx} = -\sin(x)$

Apply the product rule:

$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{dc}{dx}$

$\frac{dy}{dx} = -4\sin^2(x) + 4\cos^2(x)$

viii. $y = -2e^xx$

Let $u = -2e^x$

Let $v = x$

Differentiate both functions:

$\frac{du}{dx} = -2e^x$

$\frac{dv}{dx} = 1$

Apply the product rule:

$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$\frac{dy}{dx} = -2e^x - 2xe^x$

ix. $y = 2x^4\ln(2x)$

Let $u = 2x^4$

Let $v = \ln(2x)$

Differentiate both functions:

$$\frac{du}{dx} = 8x^3$$

$$\frac{dv}{dx} = \frac{1}{x}$$

Apply the product rule:

$$\frac{dy}{dx} = u\frac{dv}{dx} + \frac{du}{dx}$$

$$\frac{dy}{dx} = 2x^3 + 8x^3\ln(2x)$$

x. $y = 3\cos(5x)x^{-3}$

Let $u = 3\cos(5x)$

Let $v = x^{-3}$

Differentiate both functions:

$$\frac{du}{dx} = -15\sin(5x)$$

$$\frac{dv}{dx} = -3x^{-4}$$

Apply the product rule:

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{dy}{dx} = -9x^{-4}\cos(5x) - 15\sin(5x)x^{-3}$$

6. Quotient Rule

a. *i.* To calculate the derivative of a function that is the quotient of two functions. Or in other words, when we have one function of x divided by another.

ii. $\frac{dy}{dx} = \frac{v\,\frac{du}{dx} - u\,\frac{dv}{du}}{v^2}$

This means the derivative of y with respect to x is calculated from the sum of the product of u and the derivative of v with respect to x, and the product of v and the derivative of u with respect to x, all divided by v^2.

iii. Step 1: Differentiate u with respect to x, to give $\frac{du}{dx}$

Step 2: Differentiate v with respect to x, to give $\frac{dv}{dx}$

Step 3: Apply the quotient rule $\frac{dy}{dx} = \frac{v\,\frac{du}{dx} - u\,\frac{dv}{du}}{v^2}$

Answers

b. *i.* $y = \frac{5x^3}{e^x}$

Let $u = 5x^3$

Let $v = e^x$

Differentiate both functions:

$\frac{du}{dx} = 15x^2$

$\frac{dv}{dx} = e^x$

Apply the quotient rule:

$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{du}}{v^2}$

$\frac{dy}{dx} = \frac{15x^2e^x - 5x^3e^x}{(e^x)^2}$

$\frac{dy}{dx} = \frac{15x^2e^x - 5x^3e^x}{e^{2x}}$

ii. $y = \frac{-x^2}{\ln(x)}$

Let $u = -x^2$

Let $v = \ln(x)$

Differentiate both functions:

$\frac{du}{dx} = -2x$

$\frac{dv}{dx} = \frac{1}{x}$

Apply the quotient rule:

$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{du}}{v^2}$

$\frac{dy}{dx} = \frac{-2x\ln(x) + x}{(\ln(x))^2}$

iii. $y = \frac{\sin(x)}{\cos(x)}$

Let $u = \sin(x)$

Let $v = \cos(x)$

Differentiate both functions:

$\frac{du}{dx} = \cos(x)$

$\frac{dv}{dx} = -\sin(x)$

Apply the quotient rule:

$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{du}}{v^2}$

$\frac{dy}{dx} = \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)}$

We can simplify.

$\cos^2 + \sin^2 = 1$

$\frac{dy}{dx} = \frac{1}{\cos^2(x)} = \sec^2(x)$

iv. $y = \frac{-2e^{-3x}}{5\tan(x)}$

Let $u = -2e^{-3x}$

Let $v = 5\tan(x)$

Differentiate both functions:

$\frac{du}{dx} = 6e^{-3x}$

$\frac{dv}{dx} = 5\sec^2(x)$

Apply the quotient rule:

$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{du}}{v^2}$

$\frac{dy}{dx} = \frac{30\tan(x)e^{-3x} + 10\sec^2(x)e^{-3x}}{25\tan^2(x)}$

v. $y = \frac{4x}{3e^{3x}}$

Let $u = 4x$

Let $v = 3e^{3x}$

Differentiate both functions:

$$\frac{du}{dx} = 4$$

$$\frac{dv}{dx} = 9e^{3x}$$

Apply the quotient rule:

$$\frac{dy}{dx} = \frac{v\,\frac{du}{dx} - u\,\frac{dv}{du}}{v^2}$$

$$\frac{dy}{dx} = \frac{12e^{3x} - 36xe^{3x}}{(3e^{3x})^2}$$

$$\frac{dy}{dx} = \frac{12e^{3x} - 36xe^{3x}}{9e^{6x}}$$

vi. $y = \frac{12\ln(x)}{7x^3}$

Let $u = 12\ln(x)$

Let $v = 7x^3$

Differentiate both functions:

$$\frac{du}{dx} = \frac{12}{x}$$

$$\frac{dv}{dx} = 21x^2$$

Apply the quotient rule:

$$\frac{dv}{dx} = \frac{v\,\frac{du}{dx} - u\,\frac{dv}{du}}{v^2}$$

$$\frac{dv}{dx} = \frac{84x^2 - 252x^2\ln(x)}{(7x^3)^2}$$

$$\frac{dv}{dx} = \frac{84x^2 - 252x^2\ln(x)}{49x^6}$$

vii. $y = \frac{3x^{\frac{2}{3}}}{7\ln(x)}$

Let $u = 3x^{\frac{2}{3}}$

Let $v = 7\ln(x)$

Differentiate both functions:

$$\frac{du}{dx} = 2x^{-\frac{1}{3}}$$

$$\frac{dv}{dx} = \frac{7}{x}$$

Apply the quotient rule:

$$\frac{dy}{dx} = \frac{v\,\frac{du}{dx} - u\,\frac{dv}{du}}{v^2}$$

$$\frac{dy}{dx} = \frac{14\ln(x)x^{-\frac{1}{3}} - \frac{21x^{\frac{2}{3}}}{x}}{(7\ln(x))^2}$$

$$\frac{dy}{dx} = \frac{14\ln(x)x^{-\frac{1}{3}} - \frac{21x^{\frac{2}{3}}}{x}}{49\ln^2(x)}$$

viii. $y = \frac{9e^x}{8\cos(x)}$

Let $u = 9e^x$

Let $v = 8\cos(x)$

Differentiate both functions:

$$\frac{du}{dx} = 9e^x$$

$$\frac{dv}{dx} = -8\sin(x)$$

Apply the quotient rule:

$$\frac{dy}{dx} = \frac{v\,\frac{du}{dx} - u\,\frac{dv}{du}}{v^2}$$

$$\frac{dy}{dx} = \frac{72\cos(x)e^x + 72\sin(x)e^x}{(8\cos(x))^2}$$

$$\frac{dy}{dx} = \frac{72\cos(x)e^x + 72\sin(x)e^x}{64\cos^2(x)}$$

ix. $y = \frac{4\sin(x)}{9x^2}$

Let $u = 4\sin(x)$

Let $v = 9x^2$

Differentiate both functions:

$$\frac{du}{dx} = 4\cos(x)$$

$$\frac{dv}{dx} = 18x$$

Apply the quotient rule:

$$\frac{dy}{dx} = \frac{v\,\frac{du}{dx} - u\,\frac{dv}{du}}{v^2}$$

$$\frac{dy}{dx} = \frac{9x^2\cos(x) - 72x\sin(x)}{(9x^2)^2}$$

$$\frac{dy}{dx} = \frac{36x^2\cos(x) - 72x\sin(x)}{81x^4}$$

x. Note: In older versions of the workbook this question is a repeat of *ix.*. The replacement question is provided here in blue.

$y = \frac{7\tan(x)}{5\ln(2x)}$

Let $u = 7\tan(x)$

Let $v = 5\sec(x)$

Differentiate both functions:

$$\frac{du}{dx} = 7\sec^2(x)$$

$$\frac{dv}{dx} = 5\sec(x)\tan(x)$$

Apply the quotient rule:

$$\frac{dy}{dx} = \frac{v\,\frac{du}{dx} - u\,\frac{dv}{du}}{v^2}$$

$$\frac{dy}{dx} = \frac{35\sec^3(x) - 35\sec(x)\tan^2(x)}{25\sec^2(x)}$$

$$\frac{dy}{dx} = \frac{7\sec^2(x) - 7\tan^2(x)}{5\sec(x)}$$

7. Mixed Practice

a. *i.* $y = (5x^3 + 3x)^{-2}$

Chain rule and sum rule.

Let $u = 5x^3 + 3x$

So $y = u^{-2}$

Differentiate both functions:

$\frac{dy}{du} = -2u^{-3}$

$\frac{du}{dx} = 15x^2 + 3$ (sum rule)

Apply the chain rule:

$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

$\frac{dy}{dx} = -2(15x^2 + 3)u^{-3}$

Substitute back in for u:

$\frac{dy}{dx} = -2(15x^2 + 3)(5x^3 + 3x)^{-3}$

ii. $y = (4x^3 - 6x + 10)e^{2x}$

Product rule and sum rule.

Let $u = 4x^3 - 6x + 10$

Let $v = e^{2x}$

Differentiate both functions:

$\frac{du}{dx} = 12x^2 - 6$ (sum rule)

$\frac{dv}{dx} = 2e^{2x}$

Apply the product rule:

$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$\frac{dy}{dx} = 2(4x^3 - 6x + 10)e^{2x} + (12x^2 - 6)e^{2x}$

Answers

iii. $y = \frac{3}{(8x^2 - x)^2} + 5x^3$

Chain rule and sum rule.

Let $u = 8x^2 - x$

So $w = 3u^{-2}$

Differentiate both functions:

$\frac{dw}{du} = -6u^{-3}$

$\frac{du}{dx} = 16x - 1$ (sum rule)

Apply the chain rule:

$\frac{dw}{dx} = \frac{dw}{du}\frac{du}{dx}$

$\frac{dw}{dx} = -6(16x{-}1)(u)^{-3}$

Substitute back in for u:

$\frac{dw}{dx} = -6(16x - 1)(8x^2 - x)^{-3}$

Therefore,

$\frac{dy}{dx} = -6(16x - 1)(8x^2 - x)^{-3} + 15x^2$

iv. $y = \sqrt{3x^2 - 4x}$

Chain rule and sum rule.

Let $u = 3x^2 - 4x$

So $y = u^{\frac{1}{2}}$

Differentiate both functions:

$\frac{dy}{dx} = \frac{1}{2}u^{-\frac{1}{2}}$

$\frac{du}{dx} = 6x - 4$

Apply the chain rule:

$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

$\frac{dy}{dx} = \frac{1}{2}(6x - 4)u^{-\frac{1}{2}}$

Substitute back in for u:

$\frac{dy}{dx} = \frac{1}{2}(6x - 4)(3x^2 - 4x)^{-\frac{1}{2}}$

v. $y = \frac{6x^2 - 4x + 1}{5e^{-2x}}$

Quotient rule and sum rule.

Let $u = 6x^2 - 4x + 1$

Let $v = 5e^{-2x}$

Differentiate both functions:

$\frac{du}{dx} = 12x - 4$ (sum rule)

$\frac{dv}{dx} = -10e^{-2x}$

Apply the quotient rule:

$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{du}}{v^2}$

$\frac{dy}{dx} = \frac{5(12x - 4)e^{-2x} + 10(6x^2 - 4x + 1)e^{-2x}}{(5e^{-2x})^2}$

vi. $y = 7\ln(x)(5\sin(x) + 3x)$

Product rule and sum rule.

Let $u = 7\ln(x)$

Let $v = 5\sin(x) + 3x$

Differentiate both functions:

$\frac{du}{dx} = \frac{7}{x}$

$\frac{dv}{dx} = 5\cos(x) + 3$

Apply the product rule:

$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$\frac{dy}{dx} = 7\ln(x)(5\cos(x) + 3) + \frac{7}{x}(5\sin(x) + 3x)$

vii. $\sqrt[3]{7x^3 - 9x - 10}$

Chain rule and sum rule.

Let $u = 7x^3 + 9x - 10$

So $y = u^{\frac{1}{3}}$

Differentiate both functions:

$\frac{dy}{du} = \frac{1}{3}u^{-\frac{2}{3}}$

$\frac{du}{dx} = 21x^2 + 9$

Apply the chain rule:

$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

$\frac{dy}{dx} = \frac{1}{3}(21x^2 + 9)u^{-\frac{2}{3}}$

Substitute back in for u:

$\frac{dy}{dx} = \frac{1}{3}(21x^2 + 9)(7x^3 + 9x - 10)^{-\frac{2}{3}}$

viii. $\frac{9x^2 - 4x^2}{6x^2 - 3}$

Quotient rule and sum rule.

Let $u = 9x^2 - 4x^2$

$u = 5x^2$

Let $v = 6x^2 - 3$

Differentiate both functions:

$\frac{du}{dx} = 18x - 8x$

$\frac{du}{dx} = 10x$

$\frac{dv}{dx} = 19x$

Apply the quotient rule:

$\frac{dy}{dx} = \frac{v\,\frac{du}{dx} - u\,\frac{dv}{du}}{v^2}$

$\frac{dy}{dx} = \frac{(6x^2 - 3)(10x) - (5x^2)(19x)}{(6x^2 - 3)^2}$

Answers

ix. $y = \cos(9x)\sin(3x^4)$

Product rule and chain rule.

Let $u = \cos(9x)$

$v = \sin(w)$

$w = 3x^4$

Differentiate v and w:

$\frac{dv}{dw} = \cos(w)$

$\frac{dw}{dx} = 12x^3$

Apply the chain rule and substitute for w:

$\frac{dv}{dx} = \frac{dy}{dw}\frac{dw}{dx}$

$\frac{dv}{dx} = 12x^3\cos(3x^4)$

Differentiate u:

$\frac{du}{dx} = -9\sin(9x)$

Apply the product rule:

$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$\frac{dy}{dx} = -9\sin(3x^4)\sin(9x) + 12x^3\cos(9x)\cos(3x^4)$

x. $y = -\, 9(2x^2 + 7x - 12)^{\frac{2}{3}}$

Chain rule and sum rule.

Let $u = 2x^2 + 7x - 12$

So $y = -9u^{\frac{3}{2}}$

Differentiate both functions:

$\frac{dy}{du} = -\,\frac{27}{2}u^{\frac{1}{2}}$

$\frac{du}{dx} = 4x + 7$ (sum rule)

Apply the chain rule:

$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

$\frac{dy}{dx} = -\frac{27}{2}(4x + 7)u^{\frac{1}{2}}$

Substitute back in for u:

$\frac{dy}{dx} = -\frac{27}{2}(4x + 7)(2x^2 + 7x - 12)^{\frac{1}{2}}$

8. Parametric Functions

a. A function where x and y are given in terms of a third variable t. The variable t is called the parameter.

b. *i.* $\frac{dy}{dx} = \frac{dy}{dt}\,\frac{dt}{dx}$

This means the derivative of y with respect to x is calculated from the derivative of y with respect to t, multiplied by the reciprocal of the derivative of x with respect to t.

ii. Step 1: Differentiate y with respect to t, to give $\frac{dy}{dt}$

Step 2: Differentiate x with respect to t, to give $\frac{dx}{dt}$

Step 3: Take the reciprocal the derivative of x with respect to t, to give $\frac{dt}{dx}$

Step 4: Multiply the two derivatives together $\frac{dy}{dx} = \frac{dy}{dt}\,\frac{dt}{dx}$

c. *i.* $y = 2t^3$

$x = 4t^5$

Differentiate both functions:

$\frac{dy}{dt} = 6t^2$

$\frac{dx}{dt} = 20t^4$

Flip $\frac{dx}{dt}$:

$\frac{dt}{dx} = \frac{1}{20t^4}$

Multiply the derivatives:

$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx}$

$\frac{dy}{dx} = \frac{6t^2}{20t^4}$

$\frac{dy}{dx} = \frac{3}{10t^2}$

ii. $y = \cos(t)$

$x = \sin(t)$

Differentiate both functions:

$\frac{dy}{dt} = \sin(t)$

$\frac{dx}{dt} = -\cos(t)$

Flip $\frac{dx}{dt}$:

$\frac{dt}{dx} = \frac{-1}{\cos(t)}$

Multiply the derivatives:

$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx}$

$\frac{dy}{dx} = \frac{-\sin(t)}{\cos(t)}$

iii. $y = 7e^{2t}$

$x = 3\ln(t)$

Differentiate both functions:

$\frac{dy}{dt} = 14e^{2t}$

$\frac{dx}{dt} = \frac{3}{t}$

Flip $\frac{dx}{dt}$:

$\frac{dt}{dx} = \frac{t}{3}$

Multiply the derivatives:

$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx}$

$\frac{dy}{dx} = \frac{14te^{2t}}{3}$

iv. $y = 4e^{-2t}$

$x = 5t$

Differentiate both functions:

$\frac{dy}{dt} = -8e^{-2t}$

$\frac{dx}{dt} = 5$

Flip $\frac{dx}{dt}$:

$\frac{dt}{dx} = \frac{1}{5}$

Multiply the derivatives:

$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx}$

$\frac{dy}{dx} = \frac{-8e^{-2t}}{5}$

v. $y = 8\tan(t)$

$x = 12\sqrt{t}$

Differentiate both functions:

$\frac{dy}{dt} = 8\sec^2(t)$

$\frac{dx}{dt} = 6t^{-\frac{1}{2}}$

Flip $\frac{dx}{dt}$:

$\frac{dt}{dx} = \frac{1}{6t^{-\frac{1}{2}}}$

Multiply the derivatives:

$\frac{dy}{dx} = \frac{dy}{dt}\,\frac{dt}{dx}$

$\frac{dy}{dx} = \frac{8\sec^2(t)}{6t^{-\frac{1}{2}}}$

$\frac{dy}{dx} = \frac{4\sec^2(t)t^{\frac{1}{2}}}{3}$

vi. $y = 9\ln(t)$

$x = 4e^{3t}$

Differentiate both functions:

$\frac{dy}{dt} = \frac{9}{t}$

$\frac{dx}{dt} = 12e^{3t}$

Flip $\frac{dx}{dt}$:

$\frac{dt}{dx} = \frac{1}{12e^{3t}}$

Multiply the derivatives:

$\frac{dy}{dx} = \frac{dy}{dt}\,\frac{dt}{dx}$

$\frac{dy}{dx} = \frac{9}{12te^{3t}}$

9. Finding Gradients at a Point

a. The derivative describes the rate of change of the independent variable, x, with respect to the dependent variable, y. This means the derivative gives an equation for the gradient of the original function at the corresponding x values. Therefore to find a gradient at a certain point, we must differentiate, then substitute in the value of x to get our gradient.

b. *i.* $\frac{dy}{dx} = 18x - 12$

$\frac{dy}{dx} = 18(1) - 12$

$\frac{dy}{dx} = 6$

So the gradient is 6 when $x = 1$.

ii. $\frac{dy}{dx} = 4x - 1$

$\frac{dy}{dx} = 4(0) - 1$

$\frac{dy}{dx} = -1$

So the gradient is -1 when $x = 0$.

iii. $\frac{dy}{dx} = 6x + 2$

$\frac{dy}{dx} = 6(2) + 2$

$\frac{dy}{dx} = 14$

So the gradient is 14 when $x = 2$.

iv. $\frac{dy}{dx} = -5\sin(x)$

$\frac{dy}{dx} = -5\sin(6)$

$\frac{dy}{dx} = 1.40$

So the gradient is 1.40 when $x = 6$ rad.

v. $\frac{dy}{dx} = \frac{3}{x}$

$\frac{dy}{dx} = \frac{3}{4}$

So the gradient is $\frac{3}{4}$ when $x = 4$.

vi. $\frac{dy}{dx} = 24e^{2x} + 3$

$\frac{dy}{dx} = 24e^{2(1)} + 3$

$\frac{dy}{dx} = 177.34 + 3$

$\frac{dy}{dx} = 180.34$

So the gradient is 180.34 when $x = 1$.

vii. $\frac{dy}{dx} = 8\cos(4x)$

$\frac{dy}{dx} = 8\cos[4(4)]$

$\frac{dy}{dx} = -7.66$

So the gradient is –7.66 when $x = 4$ radians.

viii. $\frac{dy}{dx} = -4(4x-1)^{-2}$

$\frac{dy}{dx} = -4(4(2)-1)^{-2}$

$\frac{dy}{dx} = -0.08$

So the gradient is –0.08 when $x = 2$.

ix. $\frac{dy}{dx} = -\ 4\sin(\frac{1}{2}x + 2)$

$\frac{dy}{dx} = -\ 4\sin(\frac{1}{2}(-\ 1) + 2)$

$\frac{dy}{dx} = -3.99$

So the gradient is -3.99 when $x = -1$.

x. Note: In older versions of the workbook this question is a repeat of *vii.*. The replacement question is provided here in blue.

$y = 8\ln(x) + 7x$
at $x = 2$

$\frac{dy}{dx} = \frac{8}{x} + 7$ (using the sum rule)

$\frac{dy}{dx} = \frac{8}{2} + 7$

$\frac{dy}{dx} = 11$

So the gradient is 11 when $x = 2$.

xi. $\frac{dy}{dx} = -6xe^{-x^2}$

$\frac{dy}{dx} = -6(2)e^{-(2)^2}$

$\frac{dy}{dx} = -0.22$

So the gradient is -0.22 when $x = 2$.

xii. $\frac{dy}{dx} = \frac{10}{3}x^{-\frac{1}{3}} + 6x$

$\frac{dy}{dx} = \frac{10}{3}(3)^{-\frac{1}{3}} + 6(3)$

$\frac{dy}{dx} = 20.31$

So the gradient is 20.31 when $x = 3$.

10. Solving for x Given the Gradient

a. *i.* Note: This question has been updated in newer versions of the workbook. The replacement question is provided here in blue:

$y = 2x^2 - 12x - 10$
when $\frac{dy}{dx} = 4$

$\frac{dy}{dx} = 4x - 12$

$4 = 4x - 12$

$16 = 4x$

$x = 4$

So the gradient is 4 at $x = 4$.

ii. $\frac{dy}{dx} = 3x - 8$

$13 = 3x - 8$

$21 = 3x$

$x = 7$

So the gradient is 13 at $x = 7$.

iii. $\frac{dy}{dx} = -6x + 3$

$-15 = -6x + 3$

$-18 = -6x$

$x = 3$

So the gradient is -15 at $x = 3$.

iv. Note: This question has been updated in newer versions of the workbook. The replacement question is provided here in blue:

$y = \frac{1}{3}x^3 - \frac{11}{2}x^2 - 2$
when $\frac{dy}{dx} = -2$

$\frac{dy}{dx} = x^2 - 11$

$-2 = x^2 - 11$

$0 = x^2 - 9$

Solve the quadratic by factorisation or with the quadratic formula.

$0 = (x + 3)(x - 3)$

$x = -3$ and 3

So the gradient is -2 at both $x = 3$ and $x = -3$.

Answers

v. $\frac{dy}{dx} = -8e^{-2x}$

$-4 = -8e^{-2x}$

$\frac{1}{2} = e^{-2x}$

$\ln(\frac{1}{2}) = -2x$

$-0.69 = -2x$

$x = 0.35$

So the gradient is -4 at $x = 0.35$.

vi. $\frac{dy}{dx} = x^2 - 7x + 11$

$1 = x^2 - 7x + 11$

$0 = x^2 - 7x + 10$

Solve the quadratic by factorisation or with the quadratic formula.

$0 = (x - 2)(x - 5)$

$x = 2$ and 5

So the gradient is 1 at both $x = 2$ and $x = 5$.

vii. $\frac{dy}{dx} = -40\sin(8x)$

$-10 = -40\sin(8x)$

$\frac{1}{4} = \sin(8x)$

$0.253 = 8x$

$x = 0.032$

So the gradient is -10 at $x = 0.032$.

viii. $\frac{dy}{dx} = 3x^2 - 18x + 9$

$9 = 3x^2 - 18x + 9$

$0 = 3x^2 - 18x$

$0 = x^2 - 6x$

Solve the quadratic by factorisation or with the quadratic formula.

$0 = x(x - 6)$

$x = 0$ and 6

So the gradient is 9 at both $x = 0$ and $x = 6$.

ix. $\frac{dy}{dx} = 3\cos(3x)$

$\frac{3}{4} = 3\cos(3x)$

$\frac{1}{4} = \cos(3x)$

$1.318 = 3x$

$x = 0.439$

So the gradient is $\frac{3}{4}$ at $x = 0.439$.

x. $\frac{dy}{dx} = \frac{8x}{4x^2}$

$\frac{dy}{dx} = \frac{2}{x}$

$-1 = \frac{2}{x}$

$-x = 2$

$x = -2$

So the gradient is -1 at $x = -2$.

Answers

xi. $y = \frac{1}{2}e^{6x}$

$\frac{dy}{dx} = 6$

$\frac{dy}{dx} = 3e^{6x}$

$6 = 3e^{6x}$

$2 = e^{6x}$

$\ln(2) = 6x$

$0.69 = 6x$

$x = 0.12$

So the gradient is 6 at $x = 0.12$.

xii. $y = 2x^3 - \frac{1}{2}x^2$

$\frac{dy}{dx} = 2$

$\frac{dy}{dx} = 6x^2 - x$

$2 = 6x^2 - x$

$0 = 6x^2 - x - 2$

$0 = x^2 - \frac{1}{6}x - \frac{1}{3}$

Solve the quadratic by factorisation or with the quadratic formula.

$0 = (x + \frac{1}{2})(x - \frac{2}{3})$

$x = -\frac{1}{2}$ and $\frac{2}{3}$

So the gradient is 2 at $x = -\frac{1}{2}$ and $x = \frac{2}{3}$.

11. Normals and Tangents

a. A line that touches the curve but does not cross it. It is the line with the same gradient as the curve at the point that it touches the curve.

b. The gradient of the tangent and the gradient of the curve are equal at the point where the tangent touches the curve.

c. *i.* $\frac{dy}{dx} = 3x^2 + 6x - 5$

$\frac{dy}{dx} = 3(3)^2 + 6(3) - 5$

$\frac{dy}{dx} = 40$

$m = 40$

ii. $\frac{dy}{dx} = 16x - 10$

$\frac{dy}{dx} = 16(1) - 10$

$\frac{dy}{dx} = 6$

$m = 6$

iii. $\frac{dy}{dx} = 12x^2 - 20x$

$\frac{dy}{dx} = 12(-2)^2 + 2(-2)$

$\frac{dy}{dx} = 44$

$m = 44$

iv. $\frac{dy}{dx} = -12e^{-3x}$

$\frac{dy}{dx} = -12e^{-3(0.05)}$

$\frac{dy}{dx} = -10.3$

$m = -10.3$

v. $y = 24x^{-\frac{2}{3}} + 12x^{\frac{2}{3}}$

$x = 6$

$\frac{dy}{dx} = -16x^{-\frac{5}{3}} + 8x^{-\frac{1}{3}}$

$\frac{dy}{dx} = -16(6)^{-\frac{5}{3}} + 8(6)^{-\frac{1}{3}}$

$\frac{dy}{dx} = 3.59$

$m = 3.59$

vi. $y = 2\cos(4x - 3)$

$x = 6 \text{ rad}$

$\frac{dy}{dx} = -8\sin(4x - 3)$

$\frac{dy}{dx} = -8\sin(4(6){-}3)$

$\frac{dy}{dx} = -6.69$

$m = -6.69$

d. The equation for a line can be determined from the gradient, m, and the coordinates (x_0, y_0) by using the following equation:

$y - y_0 = m(x - x_0)$

Alternatively you can substitute the x,y coordinates into $y = mx + c$ and solve for c.

e. i. $\frac{dy}{dx} = 10x - 4$

$\frac{dy}{dx} = 10(3){-}4$

$\frac{dy}{dx} = 26$

$m = 26$

$y - 36 = 26(x - 3)$

$y - 36 = 26x - 78$

$y = 26x - 42$

ii. $\frac{dy}{dx} = 6e^{3x}$

$\frac{dy}{dx} = 6e^{3(0.25)}$

$\frac{dy}{dx} = 12.7$

$m = 12.7$

$y - 4.23 = 12.7(x - 0.25)$

$y - 4.23 = 12.7x - 3.18$

$y = 12.7x + 1.06$

iii. $\frac{dy}{dx} = 4x + 1$

$\frac{dy}{dx} = 4(2) + 1$

$\frac{dy}{dx} = 9$

$m = 9$

$y - 11 = 9(x - 2)$

$y - 11 = 9x - 18$

$y = 9x - 7$

f. A line that is perpendicular (a right angle or 90 degrees) to the tangent at a point.

g. The gradient of the normal line and the gradient of the tangent line, at any point, multiply to −1. This is the same as saying:

$n = \frac{-1}{m}$ where n is the gradient of the normal and m is the gradient.

h. *i.* $\frac{dy}{dx} = -4x + 3$

$\frac{dy}{dx} = -4(2) + 3$

$\frac{dy}{dx} = 5$

$m = \frac{-1}{5}$

ii. $\frac{dy}{dx} = 15x^2 - 16x + 25$

$\frac{dy}{dx} = 15(3) - 16(3) + 25$

$\frac{dy}{dx} = 22$

$m = \frac{-1}{22}$

iii. $\frac{dy}{dx} = -\frac{1}{2}\sin(\frac{1}{2}x)$

$\frac{dy}{dx} = -\frac{1}{2}\sin[\frac{1}{2}(0.3)]$

$\frac{dy}{dx} = -0.075$

$m = \frac{-1}{-0.075}$

$m = 13.4$

iv. $\frac{dy}{dx} = \frac{14x}{x^2}$

$\frac{dy}{dx} = \frac{14}{x}$

$\frac{dy}{dx} = \frac{14}{(1)}$

$\frac{dy}{dx} = 14$

$m = \frac{-1}{14}$

v. $y = 8x^3 - \frac{15}{2}x + \sqrt{15}$

$x = -1$

$\frac{dy}{dx} = 24x^2 - \frac{15}{2}$

$\frac{dy}{dx} = 24(-1)^2 - \frac{15}{2}$

$\frac{dy}{dx} = 16.5$

$m = \frac{-1}{16.5}$

vi. $y = 5\sin(2x - 8)$

$x = 2 \text{ rad}$

$\frac{dy}{dx} = 10\cos(2x - 8)$

$\frac{dy}{dx} = 10\cos(2(2)-8)$

$\frac{dy}{dx} = -6.54$

$m = \frac{-1}{-6.54}$

$m = 0.153$

i. *i.* $\frac{dy}{dx} = 6x^2 - 16$

$\frac{dy}{dx} = 6(3)^2 - 16$

$\frac{dy}{dx} = 38$

$m = \frac{-1}{38}$

$y - 6 = \frac{-1}{38}(x - 3)$

$y - 6 = \frac{-1}{38}x + \frac{3}{38}$

$y = \frac{-1}{38}x + \frac{231}{38}$

ii. $\frac{dy}{dx} = -\,x^{-\frac{4}{3}}$

$\frac{dy}{dx} = -(1)^{-\frac{4}{3}}$

$\frac{dy}{dx} = -1$

$m = \frac{-1}{-1}$

$m = 1$

$y - 3 = (x - 1)$

$y - 3 = x - 1$

$y = x + 2$

iii. $\frac{dy}{dx} = \frac{1}{2x}$

$\frac{dy}{dx} = \frac{1}{2(2)}$

$\frac{dy}{dx} = \frac{1}{4}$

$m = -\,4$

$y - 0.4 = -4(x - 2)$

$y - 0.4 = -4x + 8$

$y = -4x + 8.4$

12. Continuity, Limits and Differentiability

a. A function is continuous when it can be drawn with one continuous line without taking your pen off the paper.

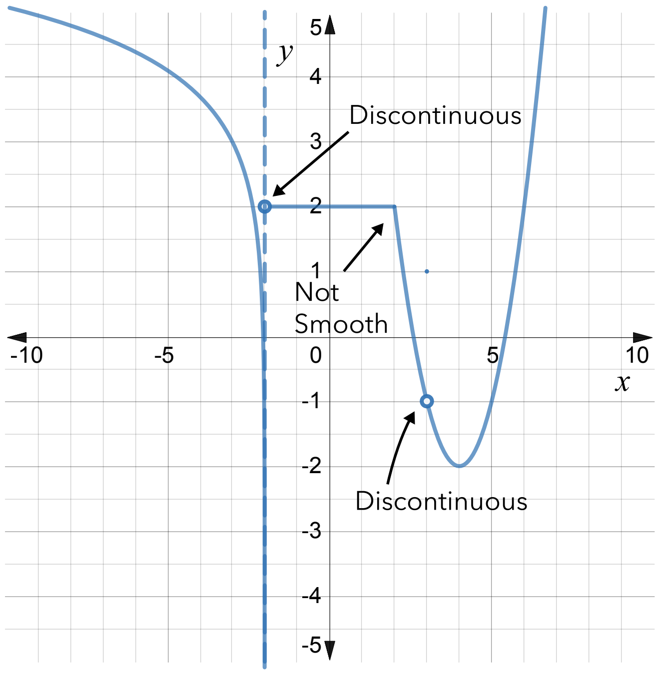
b. A function is discontinuous when there are undefined points in the function, known as removable discontinuity, or if the function jumps from one value to another, known as jump discontinuity.

c. A defined point which is an actual value given by the function.

d. An undefined point which is not an actual value given by the function.

e. A function is smooth when around some point, the function is continuous and there are no sudden changes in the gradient. This can be thought of as the function having no jagged edges.

f.



i. Discontinuous.

$x = -2$ and 3

ii. Not smooth.

$x = 2$

g. The limit of a function is the value that the function approaches as it gets closer to a particular value of x, regardless of the actual value of the function at the given value. For finite values, the function must approach the same value for both sides.

h. When the y value approached from both sides of the given x value is different. This could be due to a jump discontinuity.

i. When the y value approached from both sides of the given x value is the same. This could occur even when there is a removable discontinuity.

j. *i.* $\lim_{x \to -1} f(x) = 4$

ii. $\lim_{x \to a} g(x) = -1$

iii. $\lim_{x \to 0} h(x) = \infty$

iv. $\lim_{x \to \infty} f(x) = 0$

k. *i.* Plugging in $x = 0$ we get 0^2.
So $\lim_{x \to 0} x^2 = 0$

ii. Plugging in $x = 2$ we get $-2 + 4 = 2$.
So $\lim_{x \to 2} (-x + 4) = 2$.

iii. Plugging in $x = 3$ we get $3^2 + 4 = 13$.
So $\lim_{x \to 3} (x^2 + 4) = 13$.

iv. Plugging in $x = a$ we get $2a$.
So $\lim_{x \to a} 2x = 2a$.

v. For this it really helps to draw a graph. As x gets very very large, the value of $\frac{1}{x}$ approaches 0.

So $\lim_{x\to\infty} \frac{1}{x} = 0$.

vi. Drawing a graph would be helpful but plugging in the value works too. $|0| = 0$.

So $\lim_{x\to 0} |x| = 0$

l. *i.* $f(1) = 1$. The defined point (solid dot) at $x = 1$ is at $y = 1$.

ii. $f(4) = 2$.

iii. As x approaches 2, the function approaches $y = 2$ from both sides. Therefore, $\lim_{x\to 2} = 2$

iv. $f'(x) = 0$ means the gradient is zero. This occurs at $x = 0$.

v. $x = 1, 4$. This is where the function could not be drawn without taking your pen off the paper.

vi. $x = -2,\ 1,\ 4$. This is where the function is not continuous or there is a sharp change in the gradient.

vii. $x = -2$. This is where the function is continuous but there is a sharp change in gradient.

viii. $x = 1$. From the left-hand side, the function approaches 4 and from the right-hand side it approaches 1. Therefore as these are different, there is no limit. Note that $f(x)$ does have a limit at $x = 4$, since the function approaches 4 whether you're coming from the left or right side.

m. *i.* $f(2) = 4$

ii. Note: This question has been updated in newer versions of the workbook. The replacement question is provided here in blue:

What is the value of $f(-1)$?

$f(-1) = 2$. The defined point (solid dot) at $x = -1$ is at $y = 2$.

iii. $x = -1$. This is where the function could not be drawn without taking your pen off the paper.

iv. $x = -3, -1, 2$. This is where the function is not continuous or there is a sharp change in gradient.

v. $x = -1, 2$. This is where the function is continuous but there is a sharp change in gradient.

vi. The function is not differentiable at $x = 1, 2$.

vii. x = -1. From the left-hand side, the function approaches 2 and from the right-hand side it approaches 4. Therefore as these are different, there is no limit.

n. *i.* $f(-4) = 2$. The defined point (solid dot) at $x = -4$ is at $y = 2$.

ii. $f(4) = 0$. The defined point (solid dot) at $x = 4$ is at $y = 0$.

iii. $f'(x) = 0$ means the gradient is zero. This occurs at $x = 2$, as well as $-5 < x < -4$ and $-4 < x \leq -2$ which is a flat line.

Notice that we have excluded $x = -4$, since we can't say anything about the gradient for a single point.

iv. $x = -4, -2, 4$. This is where the function could not be drawn without taking your pen off the paper.

v. $x = -4, -2, 0, 4$. This is where the function is not continuous or there is a sharp change in gradient.

vi. $x = 0$. This is where the function is continuous but there is a sharp change in gradient.

vii. $x = -2, 4$. For $x = -2$, from the left-hand side, the function approaches 1 and from the right-hand side it approaches 3. For $x = 4$, from the left-hand side, the function approaches 0 and from the right-hand side it approaches 3. Therefore as these are different, there is no limit.

13. Features of Functions

Increasing and Decreasing Functions

a. The derivative of an increasing function will be positive. That means $\frac{dy}{dx} > 0$.

b. The derivative of a decreasing function will be negative. That means $\frac{dy}{dx} < 0$.

c. *i.* $\frac{dy}{dx} = 4(2)^3 - 3(2)$

$\frac{dy}{dx} = 26$

Increasing

ii. $\frac{dy}{dx} = -e^{-(1)} + 2$

$\frac{dy}{dx} = 1.63$

Increasing

iii. $\frac{dy}{dx} = \sin(0.5) - \cos(0.5)$

$\frac{dy}{dx} = -0.40$

Decreasing

iv. $\frac{dy}{dx} = -6x^{-4}$

$\frac{dy}{dx} = -6(1)^{-4}$

$\frac{dy}{dx} = -6$

Decreasing

v. $\frac{dy}{dx} = 3x^{-\frac{2}{3}}$

$\frac{dy}{dx} = 3(2)^{-\frac{2}{3}}$

$\frac{dy}{dx} = 1.89$

Increasing

vi. $\frac{dy}{dx} = 5\cos(x) + 3x^2$

$\frac{dy}{dx} = 5\cos(3) + 3(3)^2$

$\frac{dy}{dx} = 22.1$

Increasing

vii. $\frac{dy}{dx} = 12 - \frac{4}{x}$

$\frac{dy}{dx} = 12 - \frac{4}{(0.25)}$

$\frac{dy}{dx} = -4$

Decreasing

viii. $\frac{dy}{dx} = -9\sin(x)$

$\frac{dy}{dx} = -9\sin(10)$

$\frac{dy}{dx} = 4.90$

Increasing

ix. $\frac{dy}{dx} = -24x^7 + 6x^2$

$\frac{dy}{dx} = -24(0.8)^7 + 6(0.8)^2$

$\frac{dy}{dx} = -1.19$

Decreasing

Stationary Points

d. Minimum points, maximum points, and points of inflection.

e. The gradient at a stationary point is equal to zero.

f. The derivative is zero, as the gradient is zero. $\frac{dy}{dx} = 0$

Answers

g. *i.* $\frac{dy}{dx} = 10x + 2$

$0 = 10x + 2$

$-2 = 10x$

$x = -0.2$

ii. $\frac{dy}{dx} = 6x - 18$

$0 = 6x - 18$

$18 = 6x$

$x = 3$

iii. $\frac{dy}{dx} = x^2 - 10x + 24$

$0 = x^2 - 10x + 24$

Solve the quadratic by factorisation or with the quadratic formula.

$0 = (x - 4)(x - 6)$

$x = 4$ and 6

iv. $\frac{dy}{dx} = 3x^2 - x + 6$

$0 = 3x^2 - x + 6$

Solve the quadratic by factorisation or with the quadratic formula.

$x = 1.59$ and -1.26

v. $\frac{dy}{dx} = -\sqrt{10}\sin(x)$

$0 = -\sqrt{10}\sin(x)$

$\sin(x) = 3$

$x = \sin^{-1}(0)$

$x = 0$

vi. $\frac{dy}{dx} = 10\cos(2x)$

$0 = 10\cos(2x)$

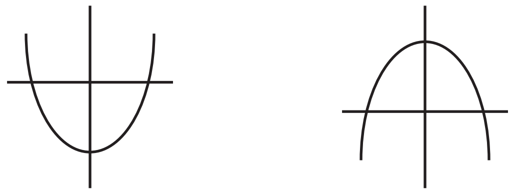
$0 = \cos(2x)$

$1.57 = 2x$

$x = 0.79$

Concavity

h.



Concave up Concave down

i. The second derivative will be positive. $\frac{d^2y}{dx^2} > 0$

j. The second derivative will be negative. $\frac{d^2y}{dx^2} < 0$

k. *i.* $\frac{d^2y}{dx^2} = 2(4)^3 - 5(4)$

$\frac{d^2y}{dx^2} = 108$

Concave up

ii. $\frac{d^2y}{dx^2} = 3(1)^2 - 10(1) + 4$

$\frac{d^2y}{dx^2} = -3$

Concave down

iii. $\frac{dy}{dx} = 24x^2 - 4x + 5$

$\frac{d^2y}{dx^2} = 48x - 4$

$\frac{d^2y}{dx^2} = 48(-0.5) - 4$

$\frac{d^2y}{dx^2} = -28$

Concave down

iv.

$\frac{dy}{dx} = 6\cos(2x)$

$\frac{d^2y}{dx^2} = -12\sin(2x)$

$\frac{d^2y}{dx^2} = -12\sin(2(3))$

$\frac{d^2y}{dx^2} = 3.35$

Concave up

v. $\frac{dy}{dx} = -5x^{-1}$

$\frac{d^2y}{dx^2} = 5x^{-2}$

$\frac{d^2y}{dx^2} = 5(3)^{-2}$

$\frac{d^2y}{dx^2} = 0.56$

Concave up

vi. $\frac{dy}{dx} = -6e^{-2x} + 3$

$\frac{d^2y}{dx^2} = 12e^{-2x}$

$\frac{d^2y}{dx^2} = 12e^{-2(0)}$

$\frac{d^2y}{dx^2} = 12$

Concave up

l. A point where the concavity changes. The value of $\frac{d^2y}{dx^2}$ is 0 at a point of inflection.

m. *i.* $\frac{d^2y}{dx^2} = 7x - 14$

$0 = 7x - 14$

$14 = 7x$

$x = 2$

ii. $\frac{d^2y}{dx^2} = 12x + 4$

$0 = 12x + 4$

$-4 = 12x$

$x = -\frac{1}{3}$

iii. $\frac{dy}{dx} = 9x^2 - 8x + 5$

$\frac{d^2y}{dx^2} = 18x - 8$

$0 = 18x - 8$

$8 = 18x$

$x = \frac{4}{9}$

iv. Note: This question has been updated in newer versions of the workbook. The replacement question is provided here in blue:

$y = x^3 - 12x^2 - 11$

$\frac{dy}{dx} = 3x^2 - 24x$

$\frac{d^2y}{dx^2} = 6x - 24$

$0 = 6x - 24$

$24 = 6x$

$x = 4$

v. $\frac{dy}{dx} = \frac{1}{3}x^3 - x^2 - 15x + 5$

$\frac{d^2y}{dx^2} = x^2 - 2x - 15$

$0 = x^2 - 2x - 15$

Solve the quadratic by factorisation or with the quadratic formula.

$0 = (x + 3)(x - 5)$

$x = -3$ and 5

vi. $\frac{dy}{dx} = \frac{4}{3}x^3 + 9x^2 + 2x$

$\frac{d^2y}{dx^2} = 4x^2 + 18x + 2$

$0 = 4x^2 + 18x + 8$

Solve the quadratic by factorisation or with the quadratic formula.

$0 = (x + \frac{1}{2})(x + 4)$

$x = -\frac{1}{2}$ and -4

vii. $\frac{dy}{dx} = \frac{1}{3}x^3 - \frac{1}{3}x^2 - 4$

$\frac{d^2y}{dx^2} = x^2 - \frac{2}{3}x$

$0 = x^2 - \frac{2}{3}x$

Solve the quadratic by factorisation or with the quadratic formula.

$0 = x(x - \frac{2}{3})$

$x = 0$ and $\frac{2}{3}$

viii. $\frac{dy}{dx} = -8\sin(4x)$

$\frac{d^2y}{dx^2} = -32\cos(4x)$

$0 = -32\cos(4x)$

$0 = \cos(4x)$

$1.57 = 4x$

$x = 0.39$

n. *i.* These are stationary points, $x = 3, -3$.

ii. This is the point of inflection, $x = 0$.

iii. This is where the graph is concave up, $x < 0$.

iv. This is where the graph is concave down, $x > 0$.

v. This is outside the turning points, $x < -3$ and $x > 3$.

vi. This is between the turning points, $-3 < x < 3$

o. *i.* Differentiating we get $f'(x) = \frac{1}{3}x^2 - 3$ and differentiating we get $f''(x) = \frac{2}{3}x$.

$\frac{2}{3}x = 0$ is only satisfied when $x = 0$.

ii. Using the same second derivative we have the inequalities $\frac{2}{3}x > 0$ when $x > 0$, and $\frac{2}{3}x < 0$ when $x < 0$.

14. Finding Maxima, Minima and Stationary Points

a. To determine whether stationary points are minima or maxima.

b. Points are taken just below and just above where the stationary point is. The first derivative is then evaluated at these points. If the curve is decreasing (has a negative slope) before, and increasing (has a positive slope) after the stationary point, it is a minima. If the curve is increasing before and decreasing after the stationary point, it is a maxima.

c. *i.* $\frac{dy}{dx} = 8x + 2$

At $x = -0.5$

$\frac{dy}{dx} = 8(-0.5)+2$

$\frac{dy}{dx} = -2$

Decreasing

At $x = 0$

$\frac{dy}{dx} = 8(0) + 2$

$\frac{dy}{dx} = 2$

Increasing

Decreasing then increasing, so the point is a minima.

ii. $\frac{dy}{dx} = 3x^2 - 16x + 5$

At $x = 0$

$\frac{dy}{dx} = 3(0)^2 - 16(0) + 5$

$\frac{dy}{dx} = 5$

Increasing

At $x = 0.5$

$\frac{dy}{dx} = 3(0.5)^2 - 16(0.5) + 5$

$\frac{dy}{dx} = -2.25$

Decreasing

Increasing then decreasing, so the point is a maxima.

iii. $\frac{dy}{dx} = 6x^2 - 6x - 10$

At $x = -1$

$\frac{dy}{dx} = 6(-1)^2 - 6(-1) - 10$

$\frac{dy}{dx} = 2$

Increasing

At $x = -0.8$

$\frac{dy}{dx} = 6(-0.8)^2 - 6(-0.8) - 10$

$\frac{dy}{dx} = -1.36$

Decreasing

Increasing then decreasing, so the point is a maxima.

iv. $\frac{dy}{dx} = 10\cos(2x)$

At $x = 0.78$ rad

$\frac{dy}{dx} = 10\cos(2(0.78))$

$\frac{dy}{dx} = 0.11$

Increasing

At $x = 0.79$

$\frac{dy}{dx} = 10\cos(2(0.79))$

$\frac{dy}{dx} = -0.09$

Decreasing

Increasing then decreasing, so the point is a maxima.

v. $\frac{dy}{dx} = 3x^2 + 4x + 1$

At $x = -0.5$

$\frac{dy}{dx} = 3(-0.5)^2 + 4(-0.5) + 1$

$\frac{dy}{dx} = -0.25$

Decreasing

At $x = 0$

$\frac{dy}{dx} = 3(0)^2 + 4(0) + 1$

$\frac{dy}{dx} = 1$

Increasing

Decreasing then increasing, so the point is a minima.

vi. $\frac{dy}{dx} = -3\sqrt{8}\sin(3x - 4)$

At $x = 1$

$\frac{dy}{dx} = -3\sqrt{8}\sin(3(1) - 4)$

$\frac{dy}{dx} = 7.1$

Increasing

At $x = 1.5$

$\frac{dy}{dx} = -3\sqrt{8}\sin(3(1.5) - 4)$

$\frac{dy}{dx} = -\ 4.1$

Decreasing

Increasing then decreasing, so the point is a maxima.

d. The second derivative is evaluated at the stationary point. If the curve is concave up (where the second derivative is positive), it is a minima. If the curve is concave down (where the second derivative is negative), it is a maxima.

e. *i.* $\frac{dy}{dx} = 18x^2 + 4x - 10$

$\frac{d^2y}{dx^2} = 36x + 4$

$\frac{d^2y}{dx^2} = 36(-\ 0.865) + 4$

$\frac{d^2y}{dx^2} = -27.14$

Concave down, so the point is a maxima

ii. $\frac{dy}{dx} = -x^{-\frac{1}{2}} + 2x$

$\frac{d^2y}{dx^2} = \frac{1}{2}x^{-\frac{3}{2}} + 2$

$\frac{d^2y}{dx^2} = \frac{1}{2}(0.63)^{-\frac{3}{2}} + 2$

$\frac{d^2y}{dx^2} = 3.0$

Concave up, so the point is a minima

iii. $\frac{dy}{dx} = 6x + 2e^{-x}$

$\frac{d^2y}{dx^2} = 6 - 2e^{-x}$

$\frac{d^2y}{dx^2} = 6 - 2e^{-(-0.619)}$

$\frac{d^2y}{dx^2} = 2.29$

Concave up, so the point is a minima

iv. $\frac{dy}{dx} = 15x^2 - 2x$

$\frac{d^2y}{dx^2} = 30x - 2$

$\frac{d^2y}{dx^2} = 30(0) - 2$

$\frac{d^2y}{dx^2} = -2$

Concave down, so the point is a maxima

v. $y = 4e^{-2x} - e^{-6x}$

$x = -0.072$

$\frac{dy}{dx} = -8e^{-2x} + 6e^{-6x}$

$\frac{d^2y}{dx^2} = 16e^{-2x} - 36e^{-6x}$

$\frac{d^2y}{dx^2} = 16e^{-2(-0.072)} - 36e^{-6(-0.072)}$

$\frac{d^2y}{dx^2} = -37.0$

Concave down, so the point is a maxima

vi. $y = 2\sin(5x + 1) + \sin(2x)$

$x = -0.072$

$\frac{dy}{dx} = 10\cos(5x + 1) + 2\cos(2x)$

$\frac{d^2y}{dx^2} = -50\sin(5x + 1) - 4\sin(2x)$

$\frac{d^2y}{dx^2} = -50\sin(5(-0.072) + 1) - 4\sin(2(-0.072))$

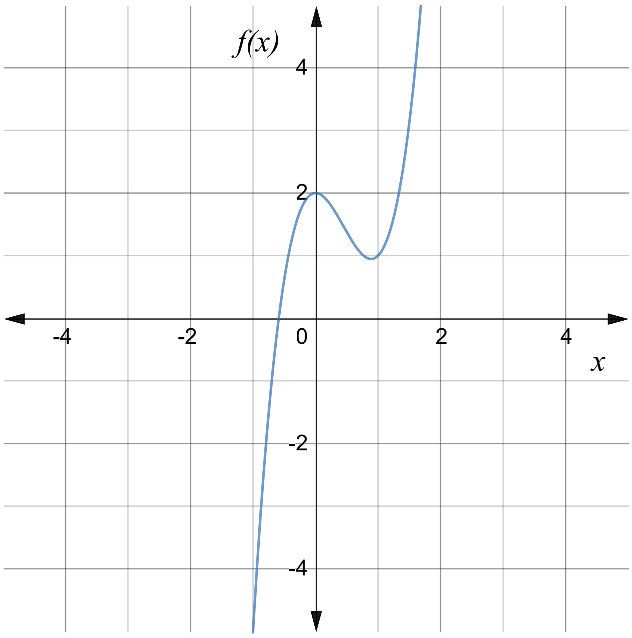
$\frac{d^2y}{dx^2} = -29.86 + 0.57$

$\frac{d^2y}{dx^2} = -29.29$

Concave down, so the point is a maxima

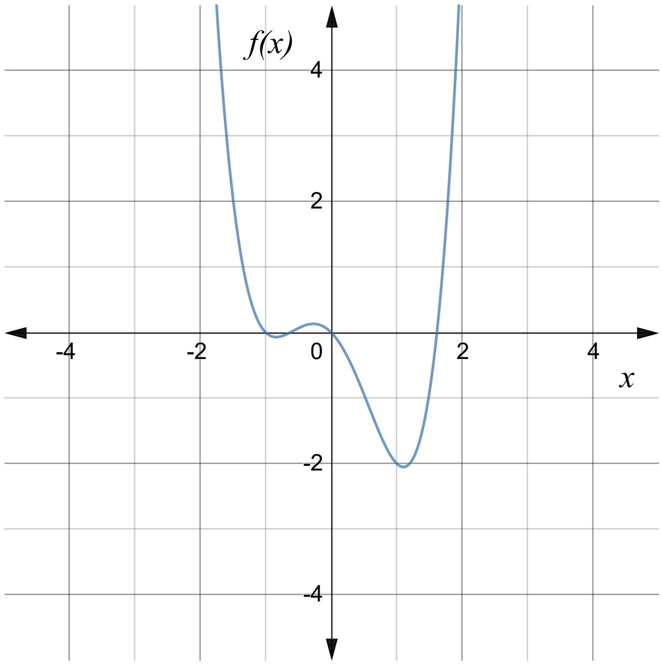
Sketching Functions

f. *i.*



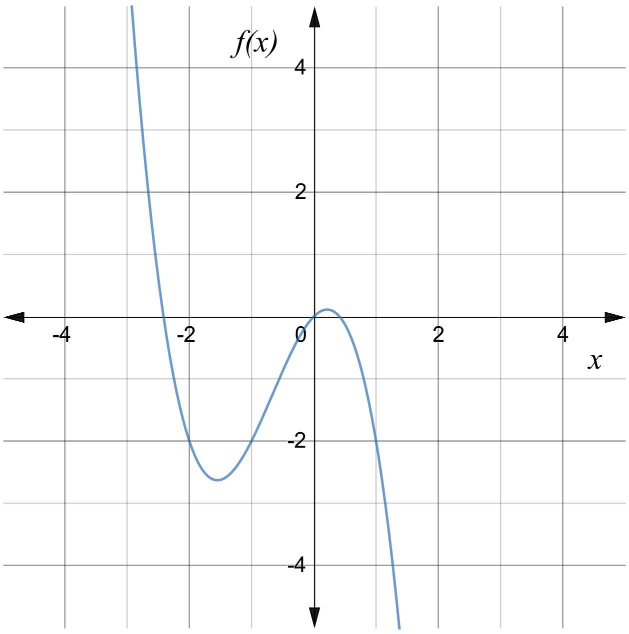
First mark the stationary points at (0,2) and (0.9,0.9). Then mark the inflection point at (0.4,1.5). Identify the regions which are increasing and decreasing, and concave up and concave down. Then draw the function accordingly.

ii. Note: In older versions of the workbook, this question is not numbered.



First mark the stationary points at (–0.8,–0.1), (–0.3,0.1) and (1.1,2.1). Then mark the inflection points at (–0.6,0) and (0.6, 1.1). Identify the regions which are increasing and decreasing, and concave up and concave down. Then draw the function accordingly.

iii. Note: In older versions of the workbook, this question is numbered *ii*.



First mark the stationary points at (–1.5,2.6) and (0.2,0.1). Then mark the inflection point at (–0.7,–1.3). Identify the regions which are increasing and decreasing, and concave up and concave down. Then draw the function accordingly.

Section Two
Exam Skills & Mixed Practice

1. Tangent and Normal Problems

a. *i.* It asks you to calculate the gradient of the tangent.

ii. $y = \frac{\sin(3x)}{4x^2 - 6x}$

iii. Need to use quotient rule as the curve equation is one function of x divided by another.

Let $u = \sin(3x)$

$v = 4x^2 - 6x$

Differentiate both functions

$\frac{du}{dx} = 3\cos(3x)$

$\frac{dv}{dx} = 8x - 6$

Apply the quotient rule

$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{du}}{v^2}$

$\frac{dy}{dx} = \frac{3(4x^2 - 6x)\cos(3x) - (8x-6)\sin(3x)}{(4x^2 - 6x)^2}$

iv. Given x value is $\frac{\pi}{6}$. First we notice that $\cos(\frac{3\pi}{6}) = \cos(\frac{\pi}{2}) = 0$.

$\frac{dy}{dx} = \frac{-(8(\frac{\pi}{2}) - 6)\sin(\frac{\pi}{2})}{(4(\frac{\pi}{6})^2 - 6(\frac{\pi}{6}))^2} = -1.57$

v. They are the same.

vi. The gradient of the tangent to the curve at $x = \frac{\pi}{6}$ is -1.57.

b. *i.* It asks you to calculate the gradient of the normal.

ii. $y = 3t^3 + 4t^2 - 10$ and $x = 2\cos(4t)$

iii. This function has been defined parametrically, so will need to use parametric differentiation.

Differentiate both functions:

$$\frac{dy}{dt} = 9t^2 + 8t$$

$$\frac{dx}{dt} = -8\sin(4t)$$

Find the negative reciprocal of $\frac{dx}{dt}$

$$\frac{dt}{dx} = \frac{-1}{-8\sin(4t)}$$

Multiply derivatives

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx}$$

$$\frac{dy}{dx} = \frac{9t^2 + 8t}{8\sin(4t)}$$

iv. Given t value is 4.

$$\frac{dy}{dx} = \frac{9(4)^2 + 8(4)}{8\sin(4(4))}$$

$$\frac{dy}{dx} = -76.41$$

v. They multiply to -1.

$$m = \frac{-1}{-76.41}$$

$$m = 0.0131$$

vi. The gradient of the normal at $t = 4$ is 0.0131.

c. *i.* It asks you to calculate the gradient of the normal.

ii. $y = \sqrt{x}\, e^{-2x}$

iii. Need to use product rule as the curve equation is one function of x multiplied by another.

Let $u = \sqrt{x}$

$v = e^{-2x}$

Differentiate both functions

$\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$

$\frac{dv}{dx} = -2e^{-2x}$

Apply the product rule

$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$\frac{dy}{dx} = -2\sqrt{x}\ e^{-2x} + \frac{e^{-2x}}{2\sqrt{x}}$

iv. Given x value is 1.

$\frac{dy}{dx} = -2\sqrt{(1)}\ e^{-2(1)} + \frac{e^{-2(1)}}{2\sqrt{(1)}}$

$\frac{dy}{dx} = -\ 0.203$

v. They multiply to -1.

$m = \frac{-1}{-0.203}$

$m = 4.9$

vi. Answer the question.

The gradient of the normal at $x = 1$ is 4.9.

d. *i.* In order for the ball to bounce back, he must be throwing the ball at a normal to the wall. We know this because we want the ball to bounce back exactly in the direction it came (so at a 90 degree angle from the wall).

ii. $y = \sqrt{3x} - 4$

iii. $\frac{dy}{dx} = \frac{1}{2}(3x)^{-\frac{1}{2}}$

iv. There is no point given for where the normal intersects the wall, so we won't determine the exact gradient until later. For now, it's okay to leave it in terms of x.

$m = \frac{-1}{\frac{1}{2}(3x)^{-\frac{1}{2}}}$

$m = -2(3x)^{\frac{1}{2}}$

v. Firstly, we want to use a formula for a linear equation since the ball will be traveling in a straight line when we look at it from above. We can use the gradient of the normal that we worked out above in this equation.

$y - y_0 = m(x - x_0)$

$y - y_0 = -2(3x)^{\frac{1}{2}}(x - x_0)$

We can use the point where Matt is standing to determine x_0 and y_0, since we know the ball must go through his coordinates (or more technically, it must start at his coordinates!)

$y - 0 = -2(3x)^{\frac{1}{2}}(x - \frac{1}{2})$

$y = -2(3x)^{\frac{1}{2}}(x - \frac{1}{2})$

vi. To find where the ball hits the wall, we must find where the path of the ball intersects with the wall curve, using our two equations from above.

$y = -2(3x)^{\frac{1}{2}}(x - \frac{1}{2})$

$y \ = \sqrt{3x} - 4$

We know the y value where the two equations meet will be the same, so set the two equations equal to each other and solve for x.

$\sqrt{3x} - 4 = -2(3x)^{\frac{1}{2}}(x - \frac{1}{2})$

$\sqrt{3x^{\frac{1}{2}}} - 4 = -2(3x)^{\frac{1}{2}}(x - \frac{1}{2})$

$\frac{\sqrt{3}x^{\frac{1}{2}}}{-2(3x)^{\frac{1}{2}}} - \frac{4}{-2(3x)^{\frac{1}{2}}} = \frac{-2(3x)^{\frac{1}{2}}(x - \frac{1}{2})}{-2(3x)^{\frac{1}{2}}}$

$\frac{-1}{2} + \frac{4}{-2(3x)^{\frac{1}{2}}} = x - \frac{1}{2}$

$\frac{4}{-2(3x)^{\frac{1}{2}}} = x$

$2 = x(3x)^{\frac{1}{2}}$

$2 = \sqrt{3x^{\frac{1}{2}}}$

$2\sqrt{3x} = x^{\frac{1}{2}}$

$x = \sqrt{\frac{2}{\sqrt{3}}}$

$x = 1.1$

Answers

Now need to find the y value from the wall curve equation.

$$y = \sqrt{3x} - 4$$

$$y = \sqrt{3(1.1)} - 4$$

$$y = -2.2$$

vii. The ball must hit the wall at (1.1, -2.2).

e. *i.* $\frac{dx}{dt} = 2\cos(t)\,\sin(t)$ by the chain rule because $x = \sin^2(t) = (\sin(t))^2$.

$\frac{dy}{dt} = -2\sin(2t)$ by the chain rule.

ii. The derivative of a parametric function is $\frac{dy}{dx} = \frac{dy}{dx} \times \frac{dx}{dt}$, which gives us:

$$\frac{dy}{dx} = -2\sin(2t)\,\frac{1}{2\cos(t)\sin(t)}$$

$$\frac{dy}{dx} = \frac{-\,2\sin(2t)}{2\cos(t)\sin(t)}$$

iii. We notice that the numerator of our derivative is a double angle, so we can rewrite the numerator of the fraction.

$-2\sin(2t) = -4\sin(t)\cos(t)$ so we rewrite our derivative.

$$\frac{dy}{dx} = \frac{-\,2\sin(2t)}{2\cos(t)\sin(t)}$$

$$\frac{dy}{dx} = \frac{-\,4\cos(t)\sin(t)}{2\cos(t)\sin(t)}$$

$$\frac{dy}{dx} = -2$$

Therefore the value of the gradient is $\frac{dy}{dx} = -2$.

f. *i.* Using the chain rule with $g(x) = \cos(x)$ and $f(g(x)) = \ln(\cos(x))$we get

$$\frac{dy}{dx} = \left(\frac{1}{\cos(x)}\right)(-\sin(x)) = \frac{-\sin(x)}{\cos(x)}$$

We notice that this is exactly the definition of the tangent function so:

$$\frac{dy}{dx} = -\tan(x)$$

ii. When $x = \frac{\pi}{4}$ we have $-\tan(\frac{\pi}{4}) = -1$

iii. We use the equation

$y - y_1 = m(x - x_1)$ where $m = -1$ so $y - y_1 = -(x - x_1)$.

We find the point (x_1, y_1), which is:

$y = \ln(\cos(\frac{\pi}{4})) + 1$

$y = \ln(\frac{\sqrt{2}}{2}) + 1$

So the equation to the tangent is

$y = -x + \frac{\pi}{4} + \ln(\frac{\sqrt{2}}{2}) + 1$

Note that the above is the final answer. It would have also been acceptable to have the answer:

$y = -x + 1.4388$

g. *i.* Using the product rule, where $u = e^x$ and $v = (x^2 - x - 1)$, then

$\frac{dy}{dx} = e^x(2x - 1) + e^x(x^2 - x - 1)$

ii. We simplify this by factorising

$\frac{dy}{dx} = e^x\ (x^2 - x - 1 + 2x - 1)$

$\frac{dy}{dx} = e^x(x^2 + x - 2)$

$\frac{dy}{dx} = e^x(x - 1)(x + 2)$

iii. The derivative is 0, which means that $\frac{dy}{dx} = 0$.

iv. When $\frac{dy}{dx} = 0$ we have $\frac{dy}{dx} = e^x(x - 1)(x + 2) = 0$.

Solving the equation, we notice that there are no values of x for which $e^x = 0$, So we can rewrite the equation as:

$(x - 1)(x + 2) = 0$

So the values of x that satisfy this equation are

$x = 1, -2$

h. *i.* If we rewrite our equation, we can use the chain rule:

$$y = (5 - x^2)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{-2x}{2\sqrt{5 - x^2}}$$

$$\frac{dy}{dx} = -\frac{x}{\sqrt{5 - x^2}}$$

ii. When $x = 1$, the value of the derivative is:

$$\frac{dy}{dx} = -\frac{1}{\sqrt{4}}$$

$$\frac{dy}{dx} = -\frac{1}{2}$$

This is the gradient of the tangent line.

iii. We have a line of the form $y = mx + c$ where $m = -\frac{1}{2}$. So:

$y = -\frac{1}{2}x + c$, when $x = 1,\ y = 2.$

Therefore we have $2 = -\frac{1}{2} + c.$

Therefore $c = \frac{5}{2}$ and the equation of the tangent line is:

$$y = -\frac{1}{2}x + \frac{5}{2}$$

iv. The tangent line intersects the x-axis when $y = 0$. So we solve the equation:

$-\frac{1}{2}x + \frac{5}{2} = 0$, rearranging we get $x = 5$.

2. Maxima, Minima, Stationary Points and Differentiability

a. *i.* Using the product rule we have $u = 2x^2$ and $v = e^x$, then $u' = 4x$ and $v' = e^x$.

The derivative is $\frac{dy}{dx} = 4xe^x + 2x^2e^x$

ii. The derivative of a function is negative when the function is decreasing.

$$\frac{dy}{dx} < 0$$

In our equation, this is represented by $4xe^x + 2x^2e^x < 0$.

iii. We can solve for stationary points (either side of which the function is either increasing or decreasing) by solving the equation:

$4xe^x + 2x^2e^x = 0$

Which we factorise to be $e^x(2x^2 + 4x) = 0$

We notice that $e^x \neq 0$, so the equation becomes: $2x^2 + 4x = 0$

Which we further factorise and solve:

$2x(x + 2) = 0$ therefore $x = 0, -2$.

iv. From the shape of the graph, the function is decreasing when $-2 < x < 0$ and increasing everywhere else.

b. *i.* The shaded circle is the value of the function. Therefore $f(-2) = 2$

ii. $x = -2, 1, 4$

iii. Note: This question has been updated in newer versions of the workbook. The replacement question is provided here in blue:

For what values of x is $f'(x) = 0$ and $f''(x) < 0$?

This is a stationary point that is concave down, which is $x = -6$.

iv. Both the left-hand and right-hand limits agree and converge to 2. Remember that it doesn't matter what f actually does at $x = 4$, as long as the line looks like it'll line up when you approach from both sides.

$\lim_{x \to 4} f(x) = 2$

v. There is no limit if the function is discontinuous, and the function approaches a different y value depending on whether you approach from above or below. This happens when $x = -2, 1$.

c. *i.* It's a bit complicated to do a product rule on something with 3 factors, so we will start by expanding the quadratic so that we only have 2 factors.

$y = e^x(x^2 - x - 6)$

This is now much easier to use the product rule on.

ii. Using the product rule, $u = e^x$ and $v = (x^2 - x - 6)$. We have

$\frac{dy}{dx} = e^x(x^2 - x - 6) + e^x(2x - 1)$

iii. The equation we are looking to solve is $\frac{dy}{dx} = 0$. So we have

$e^x(x^2 - x - 6) + e^x(2x - 1) = 0$

iv. There are no values of x that make $e^x = 0$. This means that we can easily rearrange the equation and get rid of the e^x. We get:

$e^x(x^2 - x - 6 + 2x - 1) = e^x(x^2 + x - 7) = 0$

Then we divide by e^x on both sides to get:

$x^2 + x - 7 = 0$

v. Which we solve using a calculator or the quadratic formula to get:

$x = 2.1926, -3.1926$

Or $x = \frac{-1 \pm 3\sqrt{3}}{2}$

d. *i.* We have to use both the chain rule and the product rule here, where $u = x$ and $v = \sqrt{4 - x^2}$.

Then $u' = 1$ and $v' = \frac{-2x}{2\sqrt{4 - x^2}} = \frac{-x}{\sqrt{4 - x^2}}$ by the chain rule.

So we have the derivative is:

$\frac{dy}{dx} = \sqrt{4 - x^2} - \frac{x}{\sqrt{4 - x^2}}$

ii. What is the value of the derivative when the function is stationary? Set up this equation.

The derivative of the function is equal to 0.

$\frac{dy}{dx} = 0$

Therefore, $\sqrt{4 - x^2} - \frac{x}{\sqrt{4 - x^2}} = 0$

iii. This looks a lot like a quadratic. The first things that might come to mind are squaring both sides, which will work but takes a bit of effort. Another possibility is to multiply both sides by some expression which will get rid of the square roots.

This option is a good move, and is suggested in the next question. This is a trick that often comes in handy, so make sure you understand how it works!

iv. Start by noting that $\sqrt{4-x^2} \times \sqrt{4-x^2} = 4 - x^2$. Use this fact or otherwise to rearrange for x.

We notice that we can multiply both sides by $\sqrt{4-x^2}$ to get

$$\sqrt{4-x^2}(\sqrt{4-x^2} - \frac{x^2}{2\sqrt{4-x^2}}) = \sqrt{4-x^2}$$

Which becomes:

$$4 - x^2 - x^2 = 4 - 2x^2 = 0$$

Now we just have a quadratic equation, which we rearrange to get:

$$x^2 = 2$$

v. Solving the quadratic equation gives $x = \pm\sqrt{2}$, which are the values of x where the function is stationary.

Also acceptable is $x = \pm 1.4142$ but keeping it as a surd is more accurate and preferred!

e. i. $g'(x) > 0$ means that the function is increasing.

This is $-5 < x < -2$ and $-2 < x < 2$

Or simply $-5 < x < 2,\ x \neq 2$.

ii. The value of g(–2) is the shaded circle at the point, which is 1.5

iii. Note: This question has been updated in newer versions of the workbook. The replacement question is provided here in blue:

What is the value of $\lim_{x \to 5} g(x)$? State clearly if it does not exist.

The left-hand and right-hand limits both agree, so the limit exists, and the function goes to 2. Therefore $\lim_{x \to 5} g(x) = 2$.

iv. The only point that is continuous but not differentiable is $x = -5$, because the function is not smooth there.

3. Optimisation Problems

a. i. Optimisation problems ask us to minimise or maximise something. In this case we want to determine the maximum volume of cargo, therefore it is an optimisation problem.

ii. This problem requires you to find the dimensions of the cargo that will give the maximum volume.

iii. The volume of the cargo is given by

$$V = whl$$
$$V = (2x)(2y)(15)$$
$$V = 60xy$$

iv. The tunnel is constraining the dimensions of the cargo.

v. $y = -\frac{1}{2}x^2 + \frac{9}{2}$

vi. $V = 60x(-\frac{1}{2}x^2 + \frac{9}{2})$

$V = -30x^3 + 270x$

vii. Differentiate this equation.

$$\frac{dV}{dx} = -90x^2 + 270$$

viii. The volume is maximised when the gradient and therefore the derivative is equal to 0.

$$\frac{dA}{dx} = -\ 90x^2 + 270$$

$$0 \ = -\ 90x^2 + 270$$

Use the quadratic formula or factorisation to solve for x.

$$x = -1.73 \text{ or } 1.73$$

ix. x represents half the width of the freight. –1.73m is therefore not a feasible solution as a negative length isn't possible. 1.73m is the feasible solution.

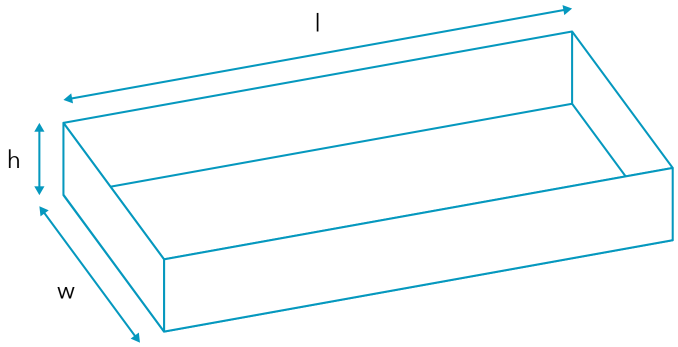
x. To determine the other variable we will need to use the constraining equation

$$y = -\frac{1}{2}(1.73)^2 + \frac{9}{2}$$

$$y = 3.00 \text{ m}$$

xi. The dimensions of the train would be 1.73 × 3.00 × 15m, giving a total volume of 77.9m^3.

b. Note: This question's diagram has been updated slightly in newer versions of the workbook. The length is now labelled as "l" instead of "R". New diagram displayed below:



i. Optimisation problems ask you to minimise or maximise something. In this case we want to determine the minimum area of plastic required, therefore it is an optimisation problem.

ii. This problem requires us to find the dimensions of the packaging that will give the minimum area and therefore the least economic and environmental cost.

iii. The area of packaging required is

$$A = 2wh + 2wl + 2hl$$

iv. The total volume of the container must be 2L.

v. $V = whl = 2000$

vi. $h = \frac{2000}{wl}$

$$A = \frac{4000}{l} + wl + \frac{4000}{w}$$

vii. The requirement for the length to be 3 times greater than the width.

$$l = 3w$$

viii. $A = \frac{4000}{3w} + 3w^2 + \frac{4000}{w}$

$$A = 3w^2 + \frac{16000}{3w}$$

ix. $\frac{dA}{dx} = 6w - \frac{16000}{3w^2}$

x. Area is maximised when the gradient and therefore the derivative is equal to 0.

$$\frac{dA}{dx} = 6w - \frac{16000}{3w^2}$$

$$0 = 6w - \frac{16000}{3w^2}$$

$$\frac{16000}{3w^2} = 6w$$

$$16000 = 18w^3$$

$$w^3 = 889$$

$$w \;\; = 9.61cm$$

ix. To determine the other variables you need to substitute our found value of x into the constraining equation.

$$l = 3w$$

$$l = 3(9.61)$$

$$l = 28.83cm$$

$$h = \frac{2000}{wl}$$

$$h = \frac{2000}{(9.61)(28.83)}$$

$$h = 0.722cm$$

xii. The dimensions of the container should be $9.61 \times 28.83 \times 0.722cm$. This might seem like a silly size but this is the size that gives the minimum area. So making a design as close to this while still functional should be the aim.

c. *i.* Optimisation problems ask you to minimise or maximise something. In this case we want to determine the shortest distance between the moon and the planet. This is effectively the minimum length, therefore it is an optimisation problem.

ii. This problem requires you to find the minimum distance between the moon and planet.

iii. We can use the Pythagorean formula for distance between two points. This gives the distance between the moon and planet, with the planet at (3, 7) and the moon at (x, y), using the equation:

$$d^2 = (3 - x)^2 + (7 - y)^2$$

We don’t need to square root this, as we can also minimise the distance squared. Either approach will give the same result if done correctly, this one is just easier so that we don't have to deal with the square root sign.

iv. The path of the moon.

v. $y = \sqrt{90000 - x^2}$

vi. $d^2 = (3 - x)^2 + (7 - \sqrt{90000 - x^2})^2$

$d^2 = x^2 - 6x + 9 + 49 - 14\sqrt{90000 - x^2} + 90000 - x^2$

$d^2 = -6x + 90058 - 14\sqrt{90000 - x^2}$

vii. Need to apply chain rule:

Let $u = 90000 - x^2$

So $w = -14\sqrt{u}$

Differentiate both functions

$\frac{dw}{du} = -7u^{-\frac{1}{2}}$

$\frac{du}{dx} = -2x$

Apply the chain rule

$\frac{dw}{dx} = \frac{dw}{du}\frac{du}{dx}$

$\frac{dw}{dx} = 14xu^{-\frac{1}{2}}$

Substitute back in for u

$\frac{dw}{dx} = 14x(90000 - x^2)^{-\frac{1}{2}}$

Therefore

$\frac{d\,d^2}{dx} = -6 + 14x(90000 - x^2)^{-\frac{1}{2}}$

Answers

viii. Distance is maximised when the gradient and therefore the derivative is equal to 0.

$\frac{d\,d^2}{dx} = -6 + 14x(90000 - x^2)^{-\frac{1}{2}}$

$0 \quad = -6 + 14x(90000 - x^2)^{-\frac{1}{2}}$

$6 \quad = 14x(90000 - x^2)^{-\frac{1}{2}}$

$\frac{6}{14} = x(90000 - x^2)^{-\frac{1}{2}}$

$\frac{6}{14(90000 - x^2)^{-\frac{1}{2}}} = x$

$\frac{6}{14}(90000 - x^2)^{\frac{1}{2}} = x$

$(\frac{6}{14}(90000 - x^2)^{\frac{1}{2}})^2 = x^2$

$\frac{9}{49}(90000 - x^2) = x^2$

$-1.184x^2 + 16530 = 0$

Use the quadratic formula to solve.

$x = 118 \text{ or } -118$

ix. x represents half the distance between the planet and moon. –118 000km is therefore not a feasible solution as a negative distance isn't possible. 118 000km is the feasible solution.

x. The closest the moon gets to the planet is 118 000km away.

d. *i.* The volume of a cylinder is given by $V = \pi r^2 h$. This is what we are trying to maximise.

ii. The volume of the cylinder is constrained by the surface area. We can calculate this by adding together the area of the curved surface and the area of the bottom of the tank.

$SA = 35 = 2\pi rh + \pi r^2$

iii. Rearranging the constraint equation for h in terms of r we have:

$35 - \pi r^2 = 2\pi rh$

$\frac{35 - \pi r^2}{2\pi r} = h$

Substituting this in to the volume equation we get:

$V = \pi r^2 \left(\frac{35 - \pi r^2}{2\pi r}\right)$

$V = \frac{35\pi r^2 - \pi^2 r^4}{2\pi r}$

$V = \frac{35}{2}r - \frac{\pi}{2}r^3$

Answers

iv. $\frac{dV}{dr} = \frac{35}{2} - \frac{3\pi}{2}r^2$

v. The volume is maximised when $\frac{dV}{dr} = 0$, so we solve the equation:

$$\frac{35}{2} - \frac{3\pi}{2}r^2 = 0$$

$$\frac{3\pi}{2}r^2 = \frac{35}{2}$$

Therefore the radius that maximises the volume is

$$r = \sqrt{\frac{35}{3\pi}} = 1.927m$$

Note that $r = -1.927m$ was another possible answer, but it doesn't make sense for a length to be negative, so our final answer was $r = 1.927m$.

4. Rate of Change Problems

Note: the following two sub-sections (4. and 5.) have changed numbering structure in newer versions of the workbook. We will include the questions with the answers to help you follow along, in case the numbering here doesn't match your workbook.

a. *i.* How do you know this problem is a rate of change problem?

Rate of change problems ask you to determine the rate something is changing at a given condition/value of something or the condition/value at which something is changing at a given rate. In this case we want to determine the rate at which concentration of ethanol is increasing 12 minutes after fermentation begins.

ii. What rate of change do you want to calculate?

We want to find the rate of change in concentration of ethanol 12 minutes after fermentation begins. $\frac{dE}{dt}$

iii. What rate of changes do you have?

We have the rate of change in concentration of ethanol per yeast cell. $\frac{dE}{dy}$

iv. What rate of change do you need? Hint: write out the chain rule to help you.

We need the rate of change in concentration of yeast cells with time.

$$\frac{dE}{dt} = \frac{dE}{dY}\frac{dY}{dt}$$

v. Can you determine a way to relate these variables?

$$Y = 3e^{5t} + 400$$

vi. Differentiate this function.

$$\frac{dY}{dt} = 15e^{0.01t}$$

vii. Apply the chain rule.

$$\frac{dE}{dt} = \frac{dE}{dY}\frac{dY}{dt}$$

$$\frac{dE}{dt} = 0.015e^{0.01t}$$

viii. Solve for the rate of change at the given conditions.

$$\frac{dE}{dt} = 0.015e^{0.01(12)}$$

$$\frac{dE}{dt} = 0.017 \text{ g/L s}$$

ix. Answer the question.

The rate of ethanol production is 0.017 g/L s.

b. *i.* How do you know this problem is a rate of change problem?

Rate of change problems ask you to determine the rate something is changing at a given condition/value of something or the condition/value at which something is changing at a given rate. In this case we want to determine the rate at which the distance between Sarah and the bird is decreasing when 10 m horizontally away.

ii. What rate of change do you want to calculate?

We want to find the rate at which the distance between Sarah and the bird is changing. $\frac{dL}{dt}$

iii. What rate of changes do you have?

We have the rate of change in horizontal distance with time (the velocity). $\frac{dx}{dt}$

iv. What rate of change do you need? Hint: write out the chain rule to help you.

We need the rate of change in distance between Sarah and the bird with the horizontal distance between them.

$$\frac{dL}{dt} = \frac{dx}{dt}\frac{dL}{dx}$$

v. Can you determine a way to relate these variables?

$$x = L\cos(\theta)$$

It is easier to differentiate this function written as $x =$ ... then invert it. You can write in terms of $L =$ ... and get the same result though.

Answers

vi. Differentiate this function.

$\frac{dx}{dL} = \cos(\theta)$

Since we were looking for $\frac{dL}{dx}$ we will invert this function.

$\frac{dL}{dx} = \frac{1}{\cos(\theta)}$

vii. Apply the chain rule.

$\frac{dL}{dt} = \frac{dx}{dt}\frac{dL}{dx}$

$\frac{dL}{dt} = \frac{11}{\cos(\theta)}$

viii. Solve for the rate of change at the given conditions.

We need to know the angle the bird is from the horizontal when it is 10 m horizontally away.

$\theta = \tan^{-1}(\frac{h}{x})$

$\theta = \tan^{-1}(\frac{5}{10})$

$\theta = 0.46 \text{ rad}$

Now we can solve for the rate of change in distance between Sarah and the bird.

$\frac{dL}{dt} = \frac{11}{\cos(0.46)}$

$\frac{dL}{dt} = 12.3\text{m/s}$

ix. Answer the question.

The rate at which the distance between Sarah and the bird is changing is 12.3m/s

c. *i.* How do you know this problem is a rate of change problem?

Rate of change problems ask you to determine the rate something is changing at a given condition/value of something or the condition/value at which something is changing at a given rate. In this case we want to determine the rate at which the surface area of the balloon is changing when the radius is 2 cm.

ii. What rate of change do you want to calculate?

We want to find the rate at which the surface area of the balloon is changing. $\frac{dS}{dt}$

iii. What rate of changes do you have?

We have the rate of change in volume of the balloon. $\frac{dV}{dt}$

iv. What rate of change do you need? Hint: write out the chain rule to help you.

We need the rate of change in surface area of the balloon with volume.

$$\frac{dS}{dt} = \frac{dV}{dt}\frac{dS}{dV}$$

v. Can you determine a way to relate these variables?

The surface area can not easily be directly related to the volume. We can instead split this up into two derivatives with another chain rule.

$$\frac{dS}{dV} = \frac{dS}{dr}\frac{dr}{dV}$$

The surface area is related to the radius by the following equation.

$$S = 2\pi rL + 2\pi r^2$$

$$S = 70\pi r + 2\pi r^2$$

The volume is related to the radius by the following equation.

$$V = \pi r^2 L$$

$$V = 35\pi r^2$$

vi. Differentiate these functions.

$$\frac{dS}{dr} = 70\pi + 4\pi r$$

$$\frac{dV}{dr} = 70\pi r$$

Since we were looking for $\frac{dr}{dV}$ we will invert this function.

$$\frac{dr}{dV} = \frac{1}{70\pi r}$$

vii. Apply the chain rule.

$$\frac{dS}{dt} = \frac{dV}{dt}\frac{dS}{dr}\frac{dr}{dV}$$

$$\frac{dS}{dt} = \frac{1700(70\pi + 4\pi r)}{70\pi(r)}$$

viii. Solve for the rate of change at the given conditions.

$$\frac{dS}{dt} = \frac{1700(70\pi + 4\pi r)}{70\pi}$$

$$\frac{dS}{dt} = \frac{1700(70\pi + 4\pi(2))}{70\pi(2)}$$

$$\frac{dS}{dt} = 947\text{cm}^2/\text{s}$$

Answers

ix. Answer the question.

The rate at which the surface area is changing is $947\text{cm}^2/\text{s}$.

5. Show and Find

Note: same with sub-section 4, again the following questions have changed numbering structure. We will include the questions with the answers to help you follow along, in case the numbering here doesn't match your workbook.

a. *i.* How do you know this is a 'show' problem?

Show problems are like differentiation proofs. They require you to evaluate both sides of an equation to show they are equal or evaluate an expression to show it is equal to something. We can tell this problem is a show problem because it includes the word 'show' and we have to prove both sides of the equation are equal.

ii. What derivatives are needed for the left-hand side?

$\frac{dy}{dx}$

iii. Determine these derivatives.

Need to use the chain rule, as this is a composite function.

Differentiate both functions:

$\frac{dy}{du} = \frac{1}{u}$

To differentiate u, you must use the product rule.

Let $w = \sin(x)$
$v = \cos(x)$

$\frac{dw}{dx} = \cos(x)$

$\frac{dv}{dx} = -\sin(x)$

Apply the product rule

$\frac{du}{dx} = v\frac{dw}{dx} + w\frac{dv}{dx}$

$\frac{du}{dx} = \cos^2(x) - \sin^2(x)$

Apply the chain rule

$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

$\frac{dy}{dx} = \frac{\cos^2(x) - \sin^2(x)}{\sin(x)\cos(x)}$

iv. Write an expression for what the left-hand side is equal to.

$$\text{LHS} = 4\frac{\cos^2(x) - \sin^2(x)}{\sin(x)\cos(x)}$$

v. What derivatives are needed for the right-hand side?

$$(1)\ \frac{d^2y}{dx^2},\ (2)\ \frac{d^2u}{dx^2},\ (3)\ \frac{du}{dx}$$

vi. Determine these derivatives.

$$(1)\ \frac{dy}{du} = \frac{1}{u} = u^{-1}$$

$$\frac{d^2y}{du^2} = -u^{-2}$$

$$\frac{d^2y}{du^2} = \frac{-1}{(\sin(x)\cos(x))^2}$$

$$(2)\ \frac{du}{dx} = = \cos^2(x) - \sin^2(x)$$

Need to use the chain rule as these are composite functions.

For $\cos^2(x)$

Let $r = p^2$, $p = \cos(x)$

$$\frac{dr}{dp} = 2p$$

$$\frac{dp}{dx} = -\sin(x)$$

$$\frac{dr}{dx} = \frac{dr}{dp}\frac{dp}{dx}$$

$$\frac{dr}{dx} = 2p\sin(x)$$

$$\frac{dr}{dx} = -2\cos(x)\sin(x)$$

Similarly for $-\sin^2(x)$

$$\frac{ds}{dx} = -2\cos(x)\sin(x)$$

Therefore

$$\frac{d^2u}{dx^2} = -4\cos(x)\sin(x)$$

(3) Already determined

vii. Write an expression for what the right-hand side is equal to.

$$\text{RHS} = \frac{4\cos(x)\sin(x)}{(\sin(x)\cos(x))^2}\ (\cos^2(x) - \sin^2(x))$$

$$\text{RHS} = 4\frac{\cos^2(x) - \sin^2(x)}{(\sin(x)\cos(x))}$$

viii. Manipulate the left or right-hand sides to make them equal.

Often once you have evaluated the left and right-hand sides they still don't look quite the same, and need factorising, expanding and simplifying to make them so. We knew what we were looking for, so they worked out to be the same right away. When you're doing these problems you may have found you had a little bit of work to go.

b. *i.* How do you know this is a 'find' problem?

A lot of NCEA problems will ask you to find things but these problems refer to finding the value(s) of constants. In this case we want to determine the value of the constant 'm' that means f(x) has a gradient of 0 at $x = 0$.

ii. What are you solving for?

The constant m.

iii. What information will help you solve for the constants?

There is one unknown, so we will need one piece of information to solve for it. The information we will use is that the gradient is 0 at $x = 0$.

iv. Differentiate the function.

$\frac{dy}{dx} = 8x - 5 + 2me^{2x}$

v. Solve for the unknown constant.

At $x = 0$, the gradient is equal to 0.

$0 = 8(0) - 5 + 2me^{2(0)}$

$0 = -5 + 2m$

$2m = 5$

$m = \frac{5}{2}$

Find the values of c and d such that $y = \frac{cx^2 - 4x}{d}$ has a stationary point at (−4,2)

i. How do you know this is a 'find' problem?

A lot of NCEA problems will ask you to find things but these problems refer to finding the value(s) of constants. In this case we want to determine the value of the constant 'c' and 'd' that means $f(x)$ has a stationary point at $(-4,2)$.

ii. What are you solving for?

The constants c and d.

iii. What information will help you solve for the constants?

There are two unknowns, so we will need two pieces of information to solve it. The information we will use is that the curve passes through the point $(-4,2)$ and that there is a stationary point at $x = -4$.

iv. Differentiate the function.

$$\frac{dy}{dx} = \frac{2c}{d}x - \frac{4}{d}$$

v. Use the stationary point to solve for one unknown constant.

At $x = -4$, the gradient is equal to 0.

$$0 = \frac{2c}{d}(-4) - \frac{4}{d}$$

$$0 = \frac{-8c}{d} - \frac{4}{d}$$

$$0 = -8c - 4$$

$$8c = -4$$

$$c = \frac{-4}{8}$$

$$c = -\frac{1}{2}$$

vi. Use the point to solve for the other unknown constant.

$$y = \frac{cx^2 - 4x}{d}$$

$$2 = \frac{(\frac{-1}{2})(-4)^2 - 4(-4)}{d}$$

$$2 = \frac{-8 + 16}{d}$$

$$2d = 8$$

$$d = 4$$

Answers

Section Three
Practice Exam

Question 1

a. Note: This question has been corrected in newer versions of the workbook. The replacement question is provided here in blue:

Differentiate $y = \frac{4}{(x^2 - 3x + 5)^4}$

Need to apply the chain rule to differentiate as it is a composite function.

Let $u = x^2 - 3x + 5$

Then $y = \frac{4}{u^4}$

Which simplifies to $y = 4u^{-4}$

Differentiate both functions:

$\frac{dy}{du} = -16u^{-5}$

$\frac{du}{dx} = 2x - 3$

Apply the chain rule:

$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

$\frac{dy}{dx} = -16(2x - 3)u^{-5}$

Substitute in for u

$\frac{dy}{dx} = -16(2x - 3)(x^2 - 3x + 5)^{-5}$

b. Need to apply quotient rule to differentiate it as it is one function of x divided by another.

Let $u = 2e^{3x}$

Let $v = 4x^2 + 7$

Differentiate both functions:

$\frac{du}{dx} = 6e^{3x}$

$\frac{dv}{dx} = 8x$

Apply the quotient rule:

$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{du}}{v^2}$

$\frac{dy}{dx} = \frac{6(4x^2 + 7)e^{3x} - 16xe^{3x}}{(4x^2 + 7)^2}$

Subtitute in x = 1:

$$\frac{dy}{dx} = \frac{6(4(1)^2 + 7)e^{3(1)} - 16(1)e^{3(1)}}{(4(1)^2 + 7)^2}$$

$$\frac{dy}{dx} = 8.3$$

c. Need to apply product rule to differentiate it as it is one function of x multiplied by another.

Let $u = 3x^2 + 8x$

Let $v = e^{2x}$

Differentiate both functions:

$$\frac{du}{dx} = 6x + 8$$

$$\frac{dv}{dx} = 2e^{2x}$$

Apply the product rule

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{dy}{dx} = 2(3x^2 + 8x)e^{2x} + (6x + 8)e^{2x}$$

$$\frac{dy}{dx} = (6x^2 + 22x + 8)e^{2x}$$

Stationary points occur where the gradient is equal to 0.

$$0 = (6x^2 + 22x + 8)e^{2x}$$

This is true when:

$$0 = 6x^2 + 22x + 8$$

Use quadratic formula:

$$x = -0.41 \text{ or } -3.26$$

d. This is a rate of change problem that requires you to find the rate at which the angle, θ, is changing, $\frac{d\theta}{dt}$.

You are given the rate at which the car is moving, $\frac{dx}{dt}$.

You need to find the rate at which the angle is changing with respect to the distance, x, $\frac{d\theta}{dt}$.

This can be shown by the chain rule:

$$\frac{d\theta}{dt} = \frac{dx}{dt}\frac{d\theta}{dx}$$

The angle can be related to x by the following equation. (It is easier to differentiate x with respect to θ, then to differentiate θ with respect to x.

$x = 50\sin(\theta)$

Differentiate this function:

$\frac{dx}{d\theta} = 50\cos(\theta)$

Apply the chain rule:

$\frac{d\theta}{dt} = \frac{dx}{dt}\frac{d\theta}{dx}$

$\frac{d\theta}{dt} = (86)(50\cos(\theta))$

In order to solve for the rate of change, you need to know θ.

$\theta = \cos^{-1}(\frac{20}{50})$

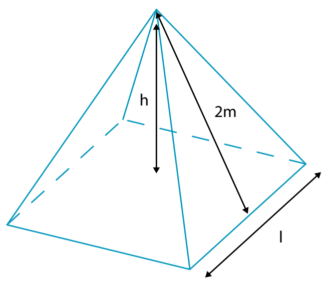
$\theta = 1.16\text{rad}$

Solve for the rate of change.

$\frac{d\theta}{dt} = (86)(50\cos(\theta))$

$\frac{d\theta}{dt} = 1720\text{rads}^{-1}$

e. Note: The diagram for this question has been slightly changed in newer versions of the workbook. The replacement diagram is provided below:



The volume of a square pyramid is given by $V = \frac{l^2h}{3}$

This is an optimization problem that requires you to find the maximum volume of a square pyramid.

$V = \frac{l^2h}{3}$

The volume is constrained by the slant length. So you need a way to relate the length, height and slant length together to substitute into the volume equation so it is only in terms of one variable, so can be differentiated.

From Pythagoras' Theorem:

$$s^2 = \left(\frac{1}{2}\right)^2 + h^2 = 2^2 = 4$$

$$l = 2\sqrt{4 - h^2}$$

Substitute this into the volume equation:

$$V = \frac{(2\sqrt{4 - h^2})^2 h}{3}$$

$$V = \frac{4(4 - h^2)h}{3}$$

$$V = \frac{16h - 4h^3}{3}$$

Differentiate:

$$\frac{dV}{dl} = \frac{16}{3} - 4h^2$$

The volume is maximised when the derivative is equal to 0.

$$0 = \frac{16}{3} - 4h^2$$

Use quadratic formula or factorise:

$$h = 1.15 \text{ m or } -1.15 \text{ m}$$

A negative height value is not a feasible solution for a height, so 1.15 m is the feasible solution.

$$V = \frac{4h - h^3}{3}$$

$$V = \frac{4(1.15) - (1.15)^3}{3}$$

$$V = 1.026 \text{ m}^2$$

Question 2

a. Need to apply the chain rule to differentiate as it is a composite function:

Let $u = 3x^2 - 9x$
Let $y = 5u^3$

Differentiate both functions:

$$\frac{dy}{du} = 15u^2$$

$$\frac{du}{dx} = 6x - 9$$

Apply the chain rule:

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

$$\frac{dy}{dx} = 15(6x - 9)u^2$$

Substitute in for u:

$$\frac{dy}{dx} = 15(6x - 9)(3x^2 - 9x)^2$$

b. *i.* $f(-3) = 4$

ii. $\lim_{x \to -3} f(x) = 2$

This is the value that is approached from both sides, not the actual value at that point.

iii.

1. $x > 0$. This is where the function is increasing.

2. $x = 0$. This is where the gradient of the function is equal to zero.

3. $x = -2$. This is where the function is continuous but not differentiable due to a sharp change in gradient.

Answers

c. Need to apply product rule to differentiate it as it is one function of x multiplied by another.

Let $u = 2\cos(3x)$
Let $v = \sin(5x)$

Differentiate both functions:

$$\frac{du}{dx} = -\ 6\sin(3x)$$

$$\frac{dv}{dx} = 5\cos(5x)$$

Apply the product rule:

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{dy}{dx} = 10\cos(3x)\cos(5x) - 6\sin(3x)\sin(5x)$$

Solve for the gradient at $x = \frac{\pi}{2}$

$$\frac{dy}{dx} = 10\cos(\frac{3\pi}{2})\cos(\frac{5\pi}{2}) - 6\sin(\frac{3\pi}{2})\sin(\frac{5\pi}{2})$$

$$\frac{dy}{dx} = 6$$

d. Differentiate the function:

$$\frac{dy}{dx} = 2e^{2x} + 9e^{-3x}$$

Differentiate again:

$$\frac{d^2y}{dx^2} = 4e^{2x} - 27e^{-3x}$$

An inflection point occurs where $\frac{d^2y}{dx^2} = 0$

$$0 = 4e^{2x} - 27e^{-3x}$$

$$4e^{2x} = 27e^{-3x}$$

$\frac{e^{2x}}{e^{-3x}} = \frac{27}{4}$ (Remember your power rules)

$$e^{5x} = \frac{27}{4}$$

$$5x = \ln(\frac{27}{4})$$

$$x = \frac{1}{5}\ln(\frac{27}{4})$$

$$x = 0.38$$

Answers

e. This is a rate of change problem that requires you to find the radius when the surface area is changing at a given rate, $\frac{dS}{dt}$

You are given the rate at which volume is changing, $\frac{dV}{dt}$.

You need to find the rate at which the surface area is changing with respect to the volume $\frac{dS}{dV}$ This can be shown by the chain rule.

$$\frac{dS}{dt} = \frac{dV}{dt}\frac{dS}{dV}$$

The surface area can not easily be directly related to the volume. We can instead split this up into two derivatives with another chain rule.

$$\frac{dS}{dV} = \frac{dS}{dr}\frac{dr}{dV}$$

The surface area is related to the radius by the following equation:

$$S = 4\pi r^2$$

The volume is related to the radius by the following equation:

$$V = \frac{4}{3}\pi r^3$$

Differentiate these functions:

$$\frac{dS}{dr} = 8\pi r$$

$$\frac{dV}{dr} = 4\pi r^2$$

Invert $\frac{dV}{dr}$ to give $\frac{dr}{dV}$

$$\frac{dr}{dV} = \frac{1}{4\pi 4^2}$$

Apply the chain rule:

$$\frac{dS}{dt} = \frac{dV}{dt}\ \frac{dS}{dr}\ \frac{dr}{dV}$$

$$\frac{dS}{dt} = -\frac{8\pi r}{4\pi r^2}$$

$$\frac{dS}{dt} = -\frac{2}{r}$$

Solve for the radius when the surface area is increasing at $0.25cm^2/s$

$$\frac{dS}{dt} = -\frac{2}{r} = -0.25$$

$$r = \frac{2}{0.25}$$

$$r = 8cm$$

Answers

Question 3

a. Need to apply quotient rule to differentiate it as it is one function of x divided by another.

Let $u = \cos(5x)$
Let $v = 4x^2 - 6x + 1$

Differentiate both functions:

$$\frac{du}{dx} = -5\sin(5x)$$

$$\frac{dv}{dx} = 8x - 6$$

Apply the quotient rule:

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{du}}{v^2}$$

$$\frac{dy}{dx} = \frac{-5(4x^2 - 6x + 1)\sin(5x) - (8x - 6)(\cos(5x))}{(4x^2 - 6x + 1)^2}$$

b. Need to apply the chain rule to differentiate as it is a composite function.

Let $u = 2x^2 - 8$
Let $y = 3\ln(u)$

Differentiate both functions:

$$\frac{dy}{du} = \frac{3}{u}$$

$$\frac{du}{dx} = 4x$$

Apply the chain rule:

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{12x}{u}$$

Substitute in for u:

$$\frac{dy}{dx} = \frac{12x}{2x^2 - 8}$$

Solve for the gradient when $x = 4$:

$$\frac{dy}{dx} = \frac{12(4)}{2(4)^2 - 8}$$

$$\frac{dy}{dx} = 2$$

c. A curve is defined parametrically by the parametric equations

$y = 5 - t^2$

$x = 7\ln(t)$

Find the gradient of the normal to this curve at the point where $t = 1$

Differentiate both functions:

$\frac{dy}{dt} = -2t$

$\frac{dx}{dt} = \frac{7}{t}$

Inverse $\frac{dx}{dt}$

$\frac{dt}{dx} = \frac{t}{7}$

Multiply derivatives

$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx}$

$\frac{dy}{dx} = \frac{-2t^2}{7}$

Calculate the gradient at $t = 1$

$\frac{dy}{dx} = \frac{-2(1)^2}{7}$

$\frac{dy}{dx} = \frac{-2}{7}$

The gradient of the normal and the curve at a given point multiply to -1. Therefore:

$m = \frac{-1}{(\frac{-2}{7})}$

$m = \frac{7}{2}$

Answers

d. This is an optimization problem that requires you to find the maximum area of a rectangle:

$A = xy$

The volume is constrained by the curve.

$y = -\frac{1}{2}x^2 + 12.5$

Substitute this into the area equation

$A = x(-\frac{1}{2}x^2 + 12.5)$

$A = -\frac{1}{2}x^3 + 12.5x$

Differentiate

$\frac{dA}{dx} = -\frac{3}{2}x^2 + 12.5$

The area is maximised when the derivative is equal to 0.

$0 = -\frac{3}{2}x^2 + 12.5$

Use quadratic formula:

$x = -2.89 \text{ or } 2.89$

A negative height value is not a feasible solution, so 2.89 is the feasible solution.

$y = -\frac{1}{2}x^2 + 12.5$

$y = -\frac{1}{2}(2.89)^2 + 12.5$

$y = 8.32$

$A = xy$

$A = (2.89)(8.32)$

$A = 24.0 \text{ units}^2$

e. This is a tangent problem

Differentiate the function:

$\frac{dy}{dx} = 4x - 6$

The gradient of the tangent is the same as the gradient of the curve:

$m = 4x - 6$

$y - y_0 = m(x - x_0)$

$y - 0 = (4x - 6)(x - 2)$

$y = 4x^2 - 14x + 12$

Substitute the one equation into the other to find the x and y coordinates where the two equations meet:

$2x^2 - 6x + 6 = 4x^2 - 14x + 12$

$0 = 2x^2 - 8x + 6$

$x = 1 \text{ or } 3$

Substitute these into the curve equation to give point A and B:

$y = 2(1)^2 - 6(1) + 6 = 2$

$A = (1,2)$

$y = 2(3)^2 - 6(3) + 6 = 6$

$B = (3,6)$

