Integration
ANSWERS

Integration Basics

Integration from Last Year
STOP AND CHECK (PAGE 5)

- $2x^3 + 0.5x^2 - 7x + c$
  -
- $\left(\frac{16}{5}\right)x^5 + 0.5^4 - 2x^2 + c$
  -
- First, we expand to get:

$$\int x^2 + x - 6dx$$

  Then we integrate:

$$\frac{1}{3}x^3 + 0.5x^2 - 6x + c$$

- In kinematics, acceleration is $\frac{dv}{dt}$ so when we integrate acceleration, we get the velocity. That is, $\int \frac{dv}{dt}$= v(t), velocity is $\frac{ds}{dt}$ so we integrate velocity to get the distance, or $\int \frac{ds}{dt}$ = s(t).

Indefinite and Definite Intervals
STOP AND CHECK (PAGE 7)

- We start by integrating:

$$\int_{1}^{2} 3x^2 - 4x + 8dx$$

To get:

$$[x^3 - 2x^2 + 8x]_{1}^{2} = ((2)^3) - 2(2)^2 + 8(2)) - ((1)^3 - 2(1)^2 + 8(1))$$

$$[x^3 - 2x^2 + 8x]_{1}^{2} = 9$$

- Again, we start by integrating:

$$\int_{2}^{4} x^2 + 6x + 2dx$$

To get:

$$[\tfrac{1}{3}x^3 + 3x^2 + 2x]_{2}^{4} = (\tfrac{1}{3}(4)^3 + 3(4)^2 + 2(4)) - (\tfrac{1}{3}(2)^3 + 3(2)^2 + 2(2))$$

$$[\tfrac{1}{3}x^3 + 3x^2 + 2x]_{2}^{4} = 58.67 \text{ (4 s.f.)}$$

Tricky Powers of x

STOP AND CHECK (PAGE 8)

- First, rearrange:

$$\int 8x^{-0.5}dx$$

Then integrate to find:

$$16x^{0.5} + c = 16\sqrt{x} + c$$

- For this one, we need to use the properties of fractions. Firstly, rearrange:

$$\int x^{\frac{3}{2}} + 2x^{-0.5}dx$$

Which integrates to:

$$\frac{2x^{\frac{5}{2}}}{5} + 4x^{0.5} + c = \frac{2\sqrt[2]{x^5}}{5} + 4\sqrt{x} + c$$

Finding c

STOP AND CHECK (PAGE 9)

- Indefinite integration.
- We need to know a coordinate. In other words, we need to know what f(x) is for some x.

Integration Basics

QUICK QUESTIONS (PAGE 9)

- For the first question, we need to remember our power rules from Level 2. A good starting point would be to rewrite the integral as:

$$\int x^{\frac{2}{5}} - x^{-12} + 7dx$$

Then finding the integral we get:

$$\frac{5x^{7/5}}{7} + \frac{1}{11x^{11}} + 7x + c$$

- We have to first evaluate the integral, which will give us:

$$f(x) = \frac{5x^{3}}{3} + 3x^{⅓} + c$$

Then, we substitute f(3) = 2 to find c, which gives us the original function:

$$f(x) = \frac{5x^{3}}{3} + 3x^{⅓} - 47.3267$$

- First, we split up the fraction to get:

$$\int x^{-2} + x^{-4}dx$$

Then we integrate to find:

$$-x^{-1} - \frac{1}{3}x^{-3} + c$$

Integrating New Functions

Exponentials

STOP AND CHECK (PAGE 10)

- The integral of e^x is e^x. However, if we have e to a power with a coefficient in it, we must divide by the derivative of the power (the reverse of chain rule), e.g.:

$$\int e^{2x}dx = 0.5e^{2x} + c$$

Remember if in doubt, you can differentiate your answer to make sure you get back to the original function.

- First, we find the derivative which is $f(x) = e^x + c$, then we substitute $x = 5$ and solve for c:

$$e^5 + c = 2$$
$$c = 2 - e^5$$
$$c = -146.413$$

So, the original function is:

$$f(x) = e^x - 146.413$$

Integrating $\frac{1}{x}$

STOP AND CHECK (PAGE 11)

- The first one doesn't require using the integral of $\frac{1}{x}$, so we just integrate as normal to get $\frac{1}{8}x^2$ + C.
- For this one we have to use the integral of $\frac{1}{x}$. We should be careful though; when first learning these it's good to rewrite the value of the fraction as the coefficient, so:

$$\int \frac{1}{2} \times \frac{1}{x}dx$$

Then we just use the rule for the natural logarithm to get:

$$\int \frac{1}{2} \times \frac{1}{x}dx = 0.5\ln|x| + c$$

Integrating New Functions

QUICK QUESTIONS (PAGE 12)

- $\int \frac{6}{x}dx = 6\ln|x| + c$
- To solve this integral, we look at the formula sheet and see that the derivative of tan(x) is sec2 (x), which tells us that:

$$\int \sec^2(x)dx = \tan(x) + c$$

Reverse Chain Rule

Introducing Substitution

STOP AND CHECK (PAGE 15)

- u is a placeholder for a substitution of a complicated part of the integral, which makes the integral much easier to solve. There is no hard and fast rule for what to choose as a good choice of u, and in many cases more than one substitution will work. We simply have to practice many cases of it to get a good feel for which substitution to make!
- Let $u = x^2$ then:

$$\frac{du}{dx} = 2x \text{ and } du = 2x dx$$

  This means the integral becomes:

$$\int \sin u \, du$$

  Once integrated, it'll look something like this:

$$-\cos(u) + c$$
$$-\cos(x^2) + c$$

- Let u be $4x^2 + 2$:

$$du = 8x dx$$

  We don't have 8x dx in our integral, but we do have 4x dx, which means we can transform our substitution to get:

$$0.5du = 4x dx$$

  Which does appear in our integral. So, now we can rewrite our integral in terms of u, to get:

$$\int 0.5\sqrt{u} du = \frac{u^{3/2}}{3} + c$$

  Then we chuck the original substitution back in to find:

$$\frac{1}{3}(4x^2 + 2)^{\frac{3}{2}} + c$$

Slightly Harder Substitutions

STOP AND CHECK (PAGE 16)

- Taking the substitution:
  - $u = x - 4$
  - $du = dx$
  - $8du = 8dx$

  The integral becomes:

$$\int 8u^3 du = 2u^4 + c$$

$$\int 8u^3 du = 2(x - 4)^4 + c$$

- Taking the substitution:
  - $u = 2x + 7$
  - $du = 2dx$
  - $3du = 6dx$

  The integral becomes:

$$\int 3u^4 du = \frac{3}{5}u^5 + c$$

$$\int 3u^4 du = \frac{3}{5}(2x + 7)^5 + c$$

- This next one is a bit tricky, let's take the substitute:
  - $u = x - 2$
  - $du = dx$
  - $x^2 = (u + 2)^2$

  The integral becomes:

$$\int (u + 2)^2 u^6 du = \int (u^2 + 4u + 4)u^6 du$$

$$\int (u + 2)^2 u^6 du = \int u^8 + 4u^7 + 4u^6 du$$

$$\int(u+2)^2u^6\text{du} = \frac{1}{9}u^9 + \frac{1}{2}u^8 + \frac{4}{7}u^7 + \text{c}$$

Fractions (Quotients)

STOP AND CHECK (PAGE 17)

- We can apply the integration rule to find:

$$4\ln|x^3 + 2x + 1| + \text{c}$$

- Again, we apply the integration rule to get:

$$0.5\ln|5x^2 + 10x| + \text{c}$$

Reverse Chain Rule

QUICK QUESTIONS (PAGE 17)

- We first make the substitution:
    - $u = x + 5$
    - $\text{du} = \text{d}x$
    - $12\text{du} = 12\text{d}x$

  The integral becomes:

$$\int 12u^{-6}\text{du} = \frac{-12}{5u^{-5}} + \text{c}$$

$$\int 12u^{-6}\text{du} = \frac{-12}{5}(x+5)^{-5} + \text{c}$$

- In this case, we take the substitution:
    - $u = x^2$
    - $\text{du} = 2x\text{d}x$

  So our integral becomes:

$$\int \text{e}^u\text{du} = \text{e}^u + \text{c}$$

$$\int e^u du = e^{x^2}$$

- The most appropriate substitution in this case is:
  - $u = x + 1$
  - $u - 1 = x$
  - $du = dx$

  The integral becomes:

$$\int \frac{u-1}{u^2} du$$

  Which we split up to get:

$$\int \frac{1}{u} - u^{-2} du = \ln|u| + u^{-1} + c$$

$$\int \frac{1}{u} - u^{-2} du = \ln|x+1| + \frac{1}{x+1} + c$$

Areas

Integrals are Areas

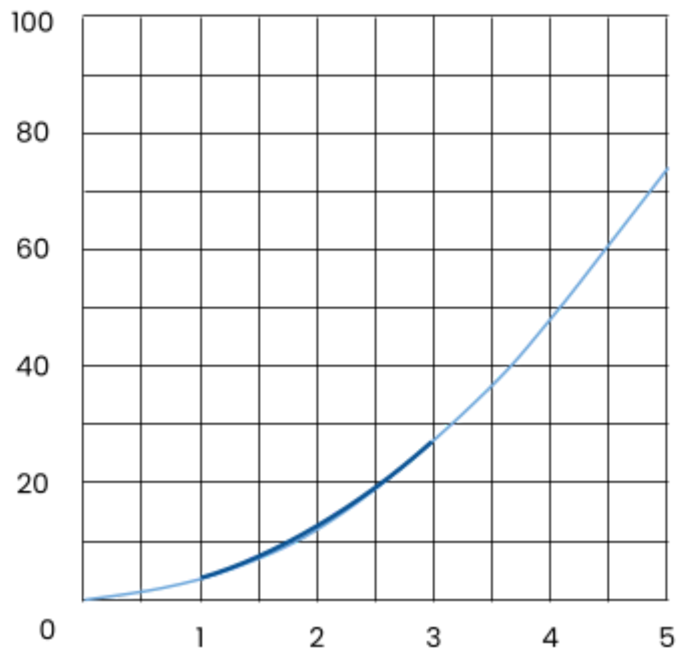
STOP AND CHECK (PAGE 19)

- Integrate distance between [0,5]. Think about how many times you need to integrate; we will need to integrate twice, using v(0) = 0 because it starts from rest and also s(0) = 0 because it starts off having covered no distance.
- Find the integral of the distance between x = 0 and x = 5 and you're done!

Definite Integrals Revisited

STOP AND CHECK (PAGE 21)

- An indefinite integral doesn't find any particular number, it gives you an equation for the original curve. A definite integral, on the other hand, finds exactly the amount of area underneath a curve, between two points. The area is marked in the dark blue line below:



Bounded Areas

STOP AND CHECK (PAGE 23)

- For a single curve, we are going to split the integral wherever y = 0, since that's usually where we go from a negative area to a positive one (or vice versa).
- Make the value of the negative area positive using absolute values because we want to find the total area enclosed by a curve.
- Add all the separate (positive) areas together. It's usually easiest to go through and work out all of the areas separately first, and then once we've got that we can add them all together.

Area Between Curves

STOP AND CHECK (PAGE 25)

- To integrate between [0, 1], we need to subtract the function on top from the one below it. In this case, it should look like:

$$\int upper - \int lower = \int_0^1 x\,dx - \int_0^1 x^2\,dx$$

$$= \int_0^1 x - x^2\,dx$$

$$= [\frac{x^2}{2} - \frac{x^3}{3}]_0^1$$

$$= \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

Area

QUICK QUESTIONS (PAGE 25)

- We have the integral:

$$v(t) = \int \sqrt{t} + 5t^{2/3} + 5$$

$$v(t) = \frac{2}{3}t^{3/2} + 3t^{5/3} + 5t + c$$

$$v(0) = 0$$

$$c = 0$$

Then, using the c = 0, we can find the distance function by integrating again:

$$s(t) = \int \frac{2}{3}t^{3/2} + 3t^{5/3}$$

$$s(t) = \frac{4}{15}t^{5/2} + \frac{9}{8}t^{8/3} + \frac{5}{2}t^2 + c$$

$$s(0) = 0$$

$$c = 0$$

Now we have to evaluate our integral between our bounds:

$$\int_0^5 \frac{4}{5}t^{5/2} + \frac{9}{8}t^{8/3} + \frac{5}{2}t^2\,dt = [\frac{8}{105}t^{7/2} + \frac{27}{88}t^{11/3} + \frac{5}{6}t^3]_0^5$$

$$= 237.605 \text{ (3 d.p.)}$$

So, in the first 5 seconds, the car travels 237.61m (2 d.p.)

- It depends on whether the area is signed or unsigned. If we want an unsigned area (which we usually do), the total area is 50 + 21 = 71. The total area above the x-axis is 50 - 21 = 29units^2.
- This is the integral:

$$\int_0^1 x - x^3 - dx = [\frac{x^2}{2} - \frac{x^4}{4}]_0^1$$

Then we evaluate:

$$\frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

Numerical Methods

Numerical Methods

QUICK QUESTIONS (PAGE 28)

- First, we determine that Δx = 0.5 since the distance between the x values is 0.5.

Now we substitute all of our values into the equation:

$$\int_2^4 f(x) = \frac{1}{3}\times 0.5\ (2.8 + 9.1 + 4(3 + 6.9) + 2(4.6)$$

$$\int_2^4 f(x) = 10.12\ (2\text{ d.p.})$$

- First, we determine that Δx = 0.1 since the distance between the x values is 0.1.

$$\int_3^6 f(x) = \frac{1}{2}\Delta x(y_0 + y_n + 2(y_1 + y_2 + \ldots + y_{n-1})dx$$

Now we substitute all of our values into the equation:

$$\int_{0.6}^{1.1} f(x) = \frac{1}{2}\times 0.1(1.2 + 2.7 + 2(1.3 + 1.5 + 1.8 + 2.2))$$

$$\int_{0.6}^{1.1} f(x) = 0.875$$

Applications of Intervals

Proportionality

STOP AND CHECK (PAGE 29)

- V = IR

Solving Differential Equations

STOP AND CHECK (PAGE 30)

- $y = x^2 + c$

Separating Variables

STOP AND CHECK (PAGE 31)

- If we first rearrange the differential equation to 'separate the variables' we get:
  - $\frac{1}{y}dy = dx$

  Then we integrate both sides:

$$\int \frac{1}{y}dy = \int 1dx$$

$$\ln|y| = x + c$$

$$y = e^{x+c}$$

So this is an exponential relationship. Yes, we can solve it.

Applications of Integrals

STOP AND CHECK (PAGE 31)

- As per the question:

$$\frac{dA}{dt} = kr$$

Where k is the constant of proportionality and r is the radius.
We are told k = 5 and r = t + 2 therefore:

$$\frac{dA}{dt} = 5(t + 2)$$

As A is the amount of water that flows through the pipe, the integral of this differential equation bound by 2 and 10 will give us our answer.

$$\int_{2}^{10} \frac{dA}{dt} dt = \int_{2}^{10} 5(t + 2) dt$$

Simplify RHS and LHS:

$$A = \int_{2}^{10} 5t + 10$$

Integrating RHS:

$$A = \frac{5t^2}{2} + 10t$$

Find the definite integral between 2 and 10:

$$\left[\frac{5t^2}{2} + 10t\right]_{2}^{10}$$

$$\left(\frac{5(10)^2}{2} + 10(10)\right) - \left(\frac{5(2)^2}{2} + 10(2)\right)$$

$$(250 + 100) - (10 + 20)$$

$$250 + 100 - 10 - 20$$

$$= 320$$

Therefore, the amount of water flowing through the pipe between 2 and 10t is $320m^2$.

