L3 COMPLEX NUMBERS
CHEAT SHEET

SURDS

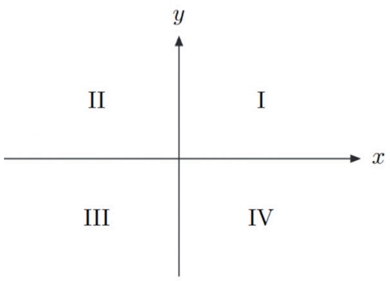
- Here are some quick rules for surds:
  - $\sqrt{a \times b} = \sqrt{a} \times \sqrt{b}$
  - $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$
  - $\frac{a}{\sqrt{b}} = \frac{a}{\sqrt{b}} \times \frac{\sqrt{b}}{\sqrt{b}} = \frac{a\sqrt{b}}{b}$

  These will help you to simplify some expressions that involve surds, more important though is how to **rationalise the denominator.**
- If you have an expression like: $\frac{a+b}{c+\sqrt{d}}$ and are told to give an answer with no surd in the denominator then you are looking to multiply by the **conjugate** of the denominator, in this case $\frac{c-\sqrt{d}}{c-\sqrt{d}}$ and this will give you an expression without the surd in the denominator. Like so:
  - $\frac{a+b}{c+\sqrt{d}} \times \frac{c-\sqrt{d}}{c-\sqrt{d}} = \frac{ac - a\sqrt{d} + bc - b\sqrt{d}}{c^2 + c\sqrt{d} - c\sqrt{d} - d} = \frac{ac - a\sqrt{d} + bc - b\sqrt{d}}{c^2 - d}$

    This looks complicated but will often simplify down very nicely in your exam!
  - Notice as well that this can be done to fractions involving complex numbers, which is also how you divide complex numbers!

COMPLEX NUMBERS

- Recall that a **complex number** is a number that is made up of two parts, a real part and an imaginary part. We'll run through the basic arithmetic, given two complex numbers:
  - $z = a + ib,\ w = x - iy$
  - Where a, b, x and y are all real numbers. Note that by convention, the *i* is written before the real coefficient, it doesn't really matter though. Be careful when following this, because the imaginary component of w is negative!
- **Argand Diagram:**
  - An argand diagram is a depiction of the *complex plane*, with the x-axis being the Real axis and the y-axis being the Complex axis, there are 4 quadrants, we'll call these I, II, III and IV and it looks like this:



  - We can plot complex numbers on this by simply using the real and complex values of the number as the x and y components (just like a normal graph).

- **Addition:**
  - Addition of complex numbers is simple – it's as easy as adding the real parts and complex parts of the complex numbers.
  - $z + w = (a + x) + i(b + y)$
    $= (a + x) + i(b - y)$
- **Subtraction:**
  - Same goes for subtraction, given the same two complex numbers, z and w, then we have,
  - $z - w = (a - x) + i(b - y)$
    $= (a - x) + i(b - y)$
- **Multiplication:**
  - Multiplying is a little harder, here you need to expand the two complex numbers (just like for expanding regular algebraic expressions) so we have:
  - $zw = (a + ib)(x + iy) = ax + iay + ibx + i^2by$
    $= (ax - by) + i(ay + bx)$
  - Notice the ax – by, this comes from the fact that $i^2 = -1$.
- **Conjugate:**
  - Before we get to division it's probably good to talk about the conjugate of a complex number, this is just the complex number with the *imaginary* part being the opposite sign. We write the conjugate as the complex number with a little bar on top of it, so we have:
  - $\overline{z} = a - ib$
  - $\overline{w} = x + iy$

  Simple as that!
- **Division:**
  - For division we have to multiply the whole thing by the conjugate of the denominator, like so:
  - $\dfrac{z}{w} = \dfrac{z}{w} \times \dfrac{\overline{w}}{\overline{w}} = \dfrac{a + ib}{x - iy} \times \dfrac{x + iy}{x + iy}$

  I'm not going to expand this out, but you could try!
- **Argument:**
  - The argument of a complex number is the angle that it makes with the real axis when plotted on an argand diagram, in other words it is the angle when written in polar form.
  - To find the argument of a complex number it's nice to plot it on an argand diagram first, notice that whenever you draw a complex number on an argand diagram that you can draw a right-angled triangle. By convention the argument of a complex number is $-\pi < \arg(z) < \pi$. So in general we calculate the angle by using the trig relationship.
  - $\tan(\theta) = \dfrac{opposite}{adjacent}$

  So, for the complex number z, this would be:
  - $\tan(\theta) = \dfrac{b}{a}$
  - $\theta = \tan^{-1}\left(\dfrac{b}{a}\right)$
  - And noticing that we bound the argument, we can compute the argument for each quadrant using this table:

Quadrant	Sign of x and y	Argument
I	$x > 0$ and $y > 0$	$\tan^{-1}(b/a)$
II	$x < 0$ and $y > 0$	$\pi + \tan^{-1}(b/a)$
III	$x < 0$ and $y < 0$	$-\pi + \tan^{-1}(b/a)$
IV	$x > 0$ and $y < 0$	$\tan^{-1}(b/a)$

- **Modulus:**

  The modulus is also fairly simple, we simply find the length of the complex number, again if we notice that when we draw a right-angled triangle then using a complex number like $w = a + ib$, the length $|w| = \sqrt{a^2 + b^2}$. Notice that this is always positive, because we cannot have a negative length.

- **Theory**

  There are a few things that you should be aware of when doing this paper, we will outline them here.

  - **Remainder Theorem:**

    Essentially, this states that for a given polynomial f(x), a linear factor (x − r) is a divisor if and only if f(r) = 0. Therefore, if the factor (x − r) has a remainder R then f(r) = R, this can be used to find unknown constants in exam questions. We can also use a technique called synthetic division to find the remainder that a factor has.

  - **Fundamental Theorem of Algebra:**

    This states that for a given polynomial of degree n, e.g. x^n, there are exactly n roots, so for a quadratic there are 2 roots, a cubic, 3 roots and so on.

  - **Conjugate Root Theorem**

    If you are given one complex root like $a + ib$ of a polynomial, one other root must be the conjugate of that root, i.e. a – ib. This is super useful for exam questions because they will often give you one complex root but not give you the other!